Massachusetts
Tests for Educator Licensure® (MTEL®)

General Curriculum (103/203)

PRACTICE TEST APPENDIX:

Multiple-Choice Question Analyses

BOOKLET 2

Mathematics Subtest

(203)

10/10

PREFACE

These analyses were written with the following guiding principles in mind:

- to give explanations for the correct responses and to provide discussions of concepts as well as procedures;
- to show different ways of solving a given problem, when appropriate;
- to explain incorrect responses in terms of typical misconceptions and error patterns in thinking about mathematical concepts;
- to explain incorrect responses in terms of any portion of correct thinking that might exist and to point out where the thought process breaks down; and
- to demonstrate ways of cross-checking work in order to find errors and to provide counterexamples.

The use of these principles results in a fairly lengthy set of analyses. In some cases, the explanations have a great degree of detail. Definitions of terms, examples of concepts, or explanations of theorems may be included in the solutions. This information may be material that you already know and understand. In cases where you understand the material, you may still find it useful to read the information to refresh your memory and strengthen your understanding of the content.

Multiple solutions have been given for some problems. In those cases, you may want to look for the solution that best matches your approach to solving the problem. Regardless of which approach is used to solve a given problem, you may find it instructive to investigate other approaches to the problem. Of course, not all possible solutions have been presented, as there are often many different ways to solve the same mathematical problem. It is likely that you may find another way of thinking about and solving a problem. In those cases, the various analyses may help you evaluate the validity of your solution.

A possible explanation of each incorrect response has been provided. While the intention is that you focus on finding a correct solution to the problem, there may be situations where looking at an incorrect response will provide further insights into the understanding necessary to give a correct response.

MULTIPLE-CHOICE QUESTION ANALYSES

1. In the number 2010, the value represented by the digit 1 is what fraction of the value represented by the digit 2?

   A. $\frac{1}{2000}$

   B. $\frac{1}{200}$

   C. $\frac{1}{20}$

   D. $\frac{1}{2}$

Correct Response B: In a place value system, the value of a digit depends on the position it occupies. For the number 2010, the value of the position of each digit is shown below.

thousands
hundreds
tens
ones
2 0 1 0

This means that the number 2010 = (2 × 1000) + (0 × 100) + (1 × 10) + (0 × 1). Therefore, the value represented by the number 2 is 2000, and the value represented by the number 1 is 10. The ratio of the value represented by the digit 1 to the value of the digit 2 is given by the following fraction.

$$\frac{\text{value represented by the digit 1}}{\text{value represented by the digit 2}} = \frac{10}{2000} = \frac{1}{200}\text{, in lowest terms}$$

Incorrect Response A: This response might have come from assuming that the value of the 1 in 2010 is actually 1 instead of 10.

Incorrect Response C: This response might have come from thinking that the 1 is in the hundreds place (perhaps from the idea that the hundredths place is two digits to the right of the decimal point), which in lowest terms would give the incorrect response $\frac{100}{2000} = \frac{1}{20}$.

Incorrect Response D: This answer disregards the place value of the digits and might have come from the informal way of saying the year 2010 as "twenty ten" and thinking $\frac{10}{20}$, which in lowest terms would be $\frac{1}{2}$.

2. If P is a positive integer, which of the following must also be a positive integer?

   A. $1 - P$

   B. $\frac{1}{P}$

   C. $\sqrt{P}$

   D. P^2

Correct Response D: The integers are the set of all positive and negative whole numbers, together with zero. They are often written as follows: {…, –3, –2, –1, 0, 1, 2, 3, …}. The positive integers are all of the *positive* numbers from this set. Since zero has neither a positive nor a negative value, zero is not included in the set of positive integers. The positive integers are often written as {1, 2, 3, …}. Notice that if P is any positive integer from the set {1, 2, 3, …}, then $P^2 = P \times P$ will also be a positive integer. It will have a positive value since a positive value times a positive value is a positive value, and it will be an integer since the product of two integers is an integer. Therefore, choice D is the correct response. For example, if $P = 11$, then $P^2 = 11^2 = 11 \times 11 = 121$, which is also a positive integer.

We can show that responses A, B, and C are all incorrect by finding a counterexample for each response. A counterexample to a statement is a single example that shows that the statement is not always true. Only one counterexample is required to show that a statement is false.

Incorrect Response A: To show that response A is incorrect, we need only find one example of a positive integer, P, for which $1 - P$ is not a positive integer. In this case, there are many counterexamples. One counterexample is $P = 1$, since if $P = 1$, then $1 - P = 0$, and 0 is not a positive integer. Notice that if $P = 2, 3, 4, \ldots$, then $1 - P$ will be a *negative* integer. Therefore, response A is incorrect.

Incorrect Response B: To show that response B is incorrect, we need only find one example of a positive integer, P, for which $\frac{1}{P}$ is not a positive integer. Notice that if $P = 2$, then $\frac{1}{P} = \frac{1}{2}$, and $\frac{1}{2}$ is not a positive integer, since it is not in the set {1, 2, 3, …}. Notice that if $P = 1$, then $\frac{1}{1} = 1$, which is a positive integer, but if $P = 2, 3, 4, \ldots$, then $\frac{1}{P}$ will be a positive rational number, but not a positive integer. A rational number is a number of the form $\frac{a}{b}$ with a and b integers and $b \neq 0$. Hence, response B is only true if $P = 1$. Since the statement must be true for all positive integers, response B is incorrect.

Incorrect Response C: To show that response C is incorrect, we need only find one example of a positive integer, P, for which $\sqrt{P}$ is not a positive integer. Notice that if $P = 2$, then $\sqrt{2} = 1.41421356\ldots$, which is not a positive integer. The value of $\sqrt{2}$ is a nonterminating, nonrepeating decimal. Nonterminating, nonrepeating decimals are also known as irrational numbers. Notice that if P is a perfect square, such as 1, 4, 9, 16, 25, …, then $\sqrt{P}$ will be a positive integer, but since the statement must be true for all positive integers, response C is incorrect.

3. According to an article in a financial journal, a certain company earned 3.85 million dollars last year. Based on this report of the company's yearly earnings, a person reading the article estimates that the company earned an average of approximately 30 thousand dollars per month. Which of the following statements best describes the reasonableness of this estimate for the company's average monthly earnings?

    A. The estimate is too low by a factor of 100.

    B. The estimate is too low by a factor of 10.

    C. The estimate is too high by a factor of 10.

    D. The estimate is too high by a factor of 100.

Correct Response B: One way to solve this problem is to divide 3.85 million dollars by 12 months. Write 3.85 million as 3,850,000. Since the number of dollars will be divided by 12 and the question is asking for an *estimate*, use compatible numbers to estimate 3,850,000 as 3,600,000, which divides evenly by 12: 3,600,000 dollars divided by 12 months equals 300,000 dollars per month. Hence, 30 thousand dollars per month is too low by a factor of 10.

Another way to solve this problem is using multiplication. If the company made $30 thousand per month and there are 12 months in the year, then it is possible to evaluate the person's estimate by finding the product of 12 and 30 thousand dollars to get 12 × $30 thousand = $360 thousand = $360,000. This estimate is too low by a factor of 10.

Incorrect Response A: If the division method had been used and 3,000,000 dollars per month had been incorrectly obtained, the estimate of 30 thousand dollars per month would seem to be too low by a factor of 100.

Incorrect Response C: If the division method had been used and 3000 dollars per month had been incorrectly obtained, the estimate of 30 thousand dollars per month would seem to be too high by a factor of 10.

Incorrect Response D: If the division method had been used and 300 dollars per month had been incorrectly obtained, the estimate of 30 thousand dollars per month would seem to be too high by a factor of 100.

4. The mean distance from the earth to the sun is approximately 93 million miles, or one astronomical unit (AU). The mean distance from Neptune to the sun is approximately 2.794×10^9 miles. What is the approximate mean distance from Neptune to the sun in astronomical units?

   A. 30 AU

   B. 300 AU

   C. 3,000 AU

   D. 30,000 AU

Correct Response A: It is given that the mean distance from the earth to the sun is 93 million miles, which is defined to be one astronomical unit (AU). It is also given that the distance from Neptune to the sun is 2.794×10^9 miles. The question asks to find how many astronomical units, or earth-sun distances, are in 2.794×10^9 miles. This number is found by dividing the distance from the sun to Neptune by the distance from the sun to the earth. To perform this division, it is helpful to use rounding techniques along with scientific notation. Rounding 93 million to 100 million gives 100×10^6 miles = 1×10^8 miles in scientific notation. Rounding 2.794 to 3 results in 3×10^9 miles. Dividing 3×10^9 miles by 1.0×10^8 miles = $\frac{3 \times 10^9}{1 \times 10^8} = 3 \times 10^{9-8} = 3 \times 10^1 = 30$. Therefore, there are approximately 30 AU (earth-sun distances) between Neptune and the sun.

Incorrect Response B: This response might have come from rounding 93 million to 100 million and incorrectly converting it to 1×10^7. The division then becomes $\frac{3 \times 10^9}{1 \times 10^7} = 3 \times 10^2 = 300$.

Incorrect Response C: This response might have come from rounding 93 million to 100 million and incorrectly converting it to 1×10^6. The division then becomes $\frac{3 \times 10^9}{1 \times 10^6} = 3 \times 10^3 = 3000$.

Incorrect Response D: This response might have come from rounding 93 million to 100 million and incorrectly converting it to 1×10^5. The division then becomes $\frac{3 \times 10^9}{1 \times 10^5} = 3 \times 10^4 = 30,000$.

5. **Use the expression below to answer the question that follows.**

$$\frac{32{,}629(484)}{306{,}751}$$

Which of the following is the best estimate of the value of the expression above?

A. 40

B. 50

C. 400

D. 500

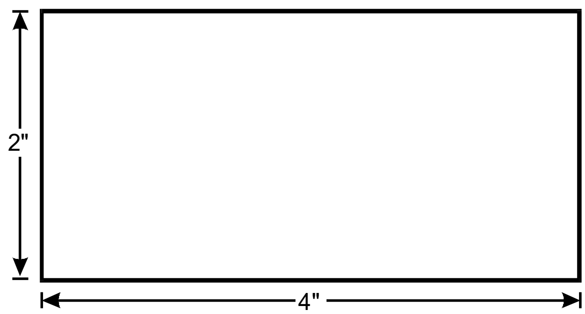
Correct Response B: To get a first estimate of the order of magnitude of the expression, round 32,629 down to 30,000, 484 up to 500, and 306,751 down to 300,000. The first estimate then becomes $\frac{30{,}000(500)}{300{,}000}$. This can also be written as $\frac{30{,}000(10)(50)}{300{,}000} = \frac{300{,}000(50)}{300{,}000} = 50$. Hence, 50 is an estimate for the value. However, since we rounded numbers up and down, and since 40, the answer in response A, is of the same order of magnitude as 50, it is best to refine the estimate to determine if the actual answer is closer to 40 or to 50. The safest approach is to find the product of 32 and 48, and then divide by 31 to better estimate the first two digits in the expression. Since 32 and 48 are smaller than the actual values, and 31 is larger, the answer will be a low estimate. $32 \times 48 = 1536$ and $1536 \div 31 \approx 49.5$. Therefore, choice B is the best estimate for the value of the expression.

Incorrect Response A: This response might have come from rounding 32,629 down to 30,000, 484 down to 400, and 306,751 down to 300,000. This estimate then becomes $\frac{30{,}000(400)}{300{,}000} = 40$. Note that this is a poor estimation technique since rounding 484 to 400 introduces a significant error.

Incorrect Response C: This response might have come from rounding 32,629 down to 30,000, 484 down to 400, and 306,751 down to 300,000, as done in incorrect response A, in addition to making an error in the order of magnitude.

Incorrect Response D: This response most likely comes from making an error in the order of magnitude when simplifying the expression shown in the correct response B.

6. **Use the diagram below to answer the question that follows.**



The measurements in the diagram above are shown rounded to the nearest whole number. Which of the following is a possible value of A, the area of the rectangle?

A. 5.0 square inches

B. 5.5 square inches

C. 11.5 square inches

D. 12.0 square inches

Correct Response B: It is given in the question that the numbers in the diagram are rounded to the nearest whole number. Let w represent the length of the smallest side of the rectangle and l represent the length of the longest side of the rectangle. Because of the rules of rounding, the smallest possible value of w is 1.5 in. Note that if w were smaller, say 1.49 in., then it would have been rounded down to 1 in. The same line of reasoning applied to l indicates that the smallest possible value of l is 3.5 in. Since the area of a rectangle A is given by $A = l \times w$, the smallest possible value for the actual area of the given rectangle is A = (1.5 in.)(3.5 in.) = 5.25 square inches.

The largest possible value of w is 2.49 in. to two decimal places, 2.499 in. to three decimal places, 2.4999 in. to four decimals places, etc. Likewise, the largest possible value of l is 4.49 in. to two decimal places, 4.499 in. to three decimal places, 4.4999 in. to four decimal places. Since these values are very close to, but less than 2.5 in. and 4.5 in., respectively, we can conclude that the area of the rectangle must always be *less than* (2.5 in.)(4.5 in) = 11.25 square inches. Therefore, the area of the rectangle must be greater than or equal to 5.25 square inches and less than 11.25 square inches ($5.25 \le A < 11.25$). Of the answer choices given, only 5.5 square inches is in this range.

Incorrect Response A: This response comes from correctly noting that the smallest possible value for the side lengths are w = 1.5 in. and l = 3.5 in., but by finding the sum of l and w instead of the product of l and w. The sum of l and w is not the area of the rectangle. The sum would be 5.0 inches, not 5.0 square inches.

Incorrect Response C: This response most likely results from incorrectly calculating the largest possible area of the rectangle, as described above.

Incorrect Response D: This response comes from using the values given in the diagram, and then finding the perimeter of (i.e., the distance around) the rectangle, instead of the area of the rectangle. The perimeter would be 12.0 inches, not 12.0 square inches.

7. **Use the table below to answer the question that follows.**

Store	Discount from Manufacturer's Recommended Price
1	$4 off the price of each game
2	30% discount on each game
3	$\frac{1}{3}$ off the price of two games
4	buy one game and get the second at half price

Samantha wants to buy two computer games, each of which has a manufacturer's recommended price of $20. She checks four different stores and finds the prices of the games discounted as shown in the table above. At which store will Samantha be able to buy the games for the least amount of money?

A. store 1

B. store 2

C. store 3

D. store 4

Correct Response C: An efficient way to answer this question is to mentally convert the discounts that each store offers to percents. Since each game cost $20, store 1 discounts $4 out of each $20, or $2 out of $10, which is a 20% discount. Since $\frac{1}{3}$ = 0.3333, the $\frac{1}{3}$ off at store 3 is a 33.33% discount. Store 4 removes $\frac{1}{2}$ of one game's price from the price of two games, which is the same as removing $\frac{1}{4}$ of a game's price from one, or a 25% discount. Therefore, with 33.33% off, store 3 (response C) offers the greatest discount.

Another way to answer this question is to calculate the price charged for the two games by each store using the information in the table. Note that the undiscounted price of a computer game is $20.00.

Store 1 takes $4 off the price of each game. One game will be sold for $20 – $4 = $16. Store 1 will charge $32 for two games. (Note that is a 20% discount.)

Store 2 discounts each game by 30%. Since 30% of $20 = 0.30 × 20 = $6, each game will cost $20 – $6 = $14. Store 2 will charge $28 for two games.

Store 3 takes $\frac{1}{3}$ of the price off the price of two games. Since two games cost $40, store 2 will charge $40 – $\frac{1}{3}$ of $40 = $40 – $\left(\frac{1}{3} \times \$40\right)$ = $40 – $13.33 = $26.67. Store 3 will charge $26.67 for two games. (Note that this is a 33.33% discount.)

Store 4 offers a "buy one and get the second at half price" discount. Therefore, store 4 sells two games for $20 + $\left(\frac{1}{2} \times \$20\right)$ = $20 + $10 = $30. (Note that this is a 25% discount.)

Comparing the offers shows that Samantha will be able to buy the games for the least amount of money at store 3. Therefore, response C is correct.

8. **Use the procedure below to answer the question that follows.**

$$\begin{aligned} n &= 0.636363... \\ 100n &= 63.636363... \end{aligned}$$

$$\begin{aligned} 100n &= 63.636363... \\ \underline{-\quad n} &= \underline{-\ 0.636363...} \\ 99n &= 63 \end{aligned}$$

The procedure above shows how to convert a repeating decimal to a fraction. If 0.12561256... is a decimal with four repeating digits, which of the following represents this decimal as a fraction?

A. $\frac{1,256}{99}$

B. $\frac{1,256}{999}$

C. $\frac{1,256}{9,999}$

D. $\frac{1,256}{99,999}$

Correct Response C: Note that the fraction for the repeating decimal 0.636363… would be $\frac{63}{99}$, found by solving $99n = 63$ for n in the procedure shown in the question. Now we will analyze the procedure used in the question. In the repeating decimal 0.636363… shown above, 63 is the part of the decimal that repeats. This is called the *period* of the decimal. The first step in converting a repeating decimal into a fraction is to multiply the decimal by a multiple of 10 large enough to shift one period to the left of the decimal point to produce a whole number equal to the period. For example, multiplying both sides of $n = 0.636363$ by 100 moves the decimal point over two places to the right, resulting in 63 to the left of the decimal point.

Let $n = 0.12561256…$. The period of the decimal 0.12561256… is 1256. To move the decimal point over 4 places to the right requires multiplication by 10,000. The procedure shown in the question is applied to this decimal.

$$\begin{aligned} n &= 0.12561256… \\ 10,000n &= 1256.12561256… \quad \text{(Multiply both sides of the equation by 10,000.)} \end{aligned}$$

Next, interchange the two equations and subtract n from $10{,}000n$. Notice that the infinitely repeating parts of the two numbers become zero due to the subtraction. This is the key reason for the success of this method.

$$\begin{array}{rl} 10{,}000n &= 1256.12561256\ldots \\ -\qquad n &= 0000.12561256\ldots \\ \hline 9{,}999n &= 1256.00000000\ldots \end{array}$$

Next, divide both sides of the equation $9{,}999n = 1256$ by 9,999 to isolate n, obtaining the final result $n = \frac{1{,}256}{9{,}999}$. This shows that the fraction $\frac{1{,}256}{9{,}999}$ is equal to 0.12561256….

Incorrect Response A: This answer most likely results from a place value error by incorrectly multiplying 0.1256126… by 100 instead of 10,000 in an effort to find 1256.12561256…, and then following the procedure shown above.

Incorrect Response B: This answer most likely results from a place value error by incorrectly multiplying 0.12561256… by 1000 instead of 10,000 in an effort to find 1256.12561256…, and then following the procedure shown above.

Incorrect Response D: This answer most likely results from a place value error by multiplying 0.12561256… by 100,000 to try to get 1256.12561256…, and then following the procedure shown above.

9. **Use the problem below to answer the question that follows.**

Given that 100 milliliters is equal to approximately 0.4 cup, 205 milliliters is equal to approximately how many cups?

Which of the following expressions models the solution to the problem above?

A. (100 – 0.4)(205)

B. 105% of 0.4

C. (205 – 100)(0.4)

D. 205% of 0.4

Correct Response D: One way to solve this problem is to use a proportion. Let *x* equal the unknown number of cups. Since the ratios of the number of milliliters to the number of cups must be equal, the following holds.

$$\frac{100 \text{ milliliters}}{0.4 \text{ cup}} = \frac{205 \text{ milliliters}}{x \text{ cups}} \quad \text{or} \quad \frac{\text{\# of milliliters}}{\text{\# of cups}} = \frac{\text{\# of milliliters}}{\text{\# of cups}}$$

Solving for *x* gives the following.

$100x = (205)(0.4)$

$x = \frac{205}{100}(0.4)$

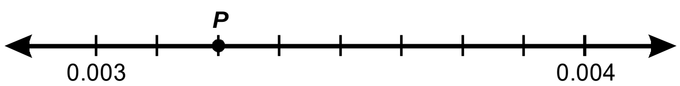
Now, look at the answer choices and notice that this is equivalent to response D, since $\frac{205}{100}$ is equal to 205 per hundred or 205%. Therefore, the expression $\frac{205}{100}(0.4) = 205\%$ of 0.4.

Incorrect Response A: This response comes from incorrectly assuming that the number of cups represented by one milliliter is equal to the difference between 100 and 0.4, so (100 – 0.4)(205) would represent the number of cups in 205 milliliters.

Incorrect Response B: This response comes from incorrectly assuming that the number of cups represented by 205 millimeters is equal to $\frac{205 - 100}{100}(0.4) = \frac{105}{100}(0.4) = 105\%$ of 0.4.

Incorrect Response C: This response comes from incorrectly assuming that the number of cups represented by 205 milliliters can be found by calculating the product of 0.4 and the difference of 205 and 100.

10. **Use the number line below to answer the question that follows.**



What number is represented by point *P* on the number line above?

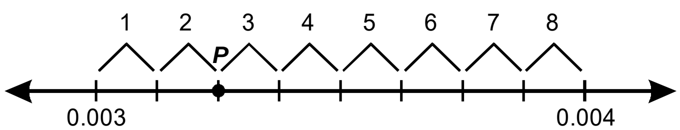
A. 0.0032

B. 0.00325

C. 0.0034

D. 0.00345

Correct Response B: A number line is a representation of the real number system in which each real number corresponds to a point on the line. In the number line shown, the smaller number represented is 0.003, and the larger number is 0.004. Notice that there are 8 equal spaces between the two numbers.



Also, notice that the length of the line segment connecting 0.004 and 0.003 is $0.004 - 0.003 = 0.001$. Since there are 8 equal spaces between the two points, the distance between any two tick marks is $0.001 \div 8 = 0.000125$. Since point *P* is located 2 tick marks to the right of 0.003, the coordinate of point *P* is $0.003 + 2(0.000125) = 0.003 + 0.00025 = 0.00325$.

It is important to point out that it is a common mistake is to assume that the tick marks between two points on a number line will always define 10 equal spaces. As a teacher, and while taking this test, it is necessary to avoid these mistakes by paying careful attention to details when doing mathematical work.

Incorrect Response A: This response most likely comes from not counting the number of spaces between 0.003 and 0.004 and assuming that there are 10 spaces between the two points, which would result in an incorrect distance of 0.0001 between tick marks. Since *P* is located 2 tick marks to the right of 0.003, the coordinate of point *P* would be $0.003 + 2(0.0001) = 0.003 + 0.0002 = 0.0032$.

Incorrect Response C: This response most likely comes from noticing that point *P* is one-fourth of the distance from 0.003 and 0.004 and then incorrectly writing one-fourth of 0.001 as 0.0004.

Incorrect Response D: This response most likely comes from noticing that point *P* is less than half the distance from 0.003 and 0.004 and then estimating point *P* as 0.00345.

11. A book distributor is trying to divide an order of textbooks into equally sized groups for shipping in cartons. The textbooks can be divided into groups of 12, groups of 15, or groups of 18, with no books left over. Which of the following inequalities is satisfied if *N* is the smallest possible total number of textbooks?

    A. $100 \leq N < 150$

    B. $150 \leq N < 200$

    C. $200 \leq N < 250$

    D. $250 \leq N < 300$

Correct Response B: The distributor has a number of textbooks, *N*. Since the number of books can be separated into piles of 12, 15, and 18 with no books left over, *N* is divisible by 12, 15, and 18. The question asks for the smallest number that is divisible by 12, 15, and 18. Notice that this is equivalent to finding the least common multiple (LCM) of 12, 15, and 18. One way to find the least common multiple is to make a list of the set of multiples for each number, as follows:

Multiples of 12: {12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216, 228, 240, 252, 264, 276, 288, 300, …}

Multiples of 15: {15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, 225, 240, 255, 270, 285, 300, …}

Multiples of 18: {18, 36, 54, 72, 90, 108, 126, 144, 162, 180, 198, 216, 234, 252, 270, 288, 306, …}

The smallest number that occurs in each of the three lists of multiples is 180; hence, 180 is the LCM of 12, 15, and 18. That is, 180 is the smallest number divisible by all three numbers, 12, 15, and 18, and 180 falls in the range $150 \leq N < 200$.

Another method to find the least common multiple is to find the prime factorization of each number and then determine the smallest number that has 12, 15, and 18 as divisors.

$12 = 2 \cdot 2 \cdot 3$
$15 = 3 \cdot 5$
$18 = 2 \cdot 3 \cdot 3$

Since *N* must be divisible by 12, *N* must have factors of 2, 2, and 3. Since *N* must be divisible by 15, *N* must have factors of 3 and 5, but only the 5 is needed. Since *N* must be divisible by 18, it must have factors of 2, 3, and 3, so another 3 is needed.

Since *N* has to be the smallest number that is divisible by each of these three numbers, $N = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 3 = 180$. The number 180 is divisible by 12 (i.e., by $2 \cdot 2 \cdot 3$), by 15 (i.e., by $3 \cdot 5$), and by 18 (i.e., by $2 \cdot 3 \cdot 3$). From the answer choices, note that 180 falls in the range $150 \leq N < 200$.

Incorrect Response A: This response may have come from observing that 144 is in both the list of multiples of 12 and the list of multiples of 18, and fits in the range of values given in this response. However, 144 is not in the list of multiples of 15.

Incorrect Response C: This response may have come from assuming that the LCM = 12 • 18 = 2 • 2 • 3 • 2 • 3 • 3 = $2^3 \cdot 3^3$ = 216, or from observing that 216 is in both the list of multiples of 12 and the list of multiples of 18, and fits in the range of values given in this response. But, 216 is not in the list of multiples of 15.

Incorrect Response D: This response may have come from assuming that the LCM = 15 • 18 = 3 • 5 • 2 • 3 • 3 = $2 \cdot 3^3 \cdot 5$ = 270, or from observing that 270 is both in the list of multiples of 15 and 18, and fits in the range of values given in this response. But, 270 is not a multiple of 12.

12. The prime factorization of a natural number n can be written as $n = pr^2$ where p and r are distinct prime numbers. How many factors does n have, including 1 and itself?

A. 3

B. 4

C. 5

D. 6

Correct Response D: Since we are given the prime factorization of n as p, r, and r, we can determine all of the factors of n. For example, suppose $n = 18$. The prime factorization of 18 is $18 = 2 \cdot 3 \cdot 3 = 2 \cdot 3^2$. Factors are 1 and itself ($2 \cdot 3 \cdot 3$), along with every unique combination of factors, as shown below.

1

2

3

$2 \cdot 3$

$3 \cdot 3$

$2 \cdot 3 \cdot 3$

There are 6 factors of 18. The set of factors is {1, 2, 3, 6, 9, 18}. It is interesting to note that the list can be thought of as pairs 1 and 18, 2 and 9, and 3 and 6.

Using the same method with $n = pr^2 = p \cdot r \cdot r$, we can list all of the factors of n.

1

p

$p \cdot r$

$r \cdot r$

$p \cdot r \cdot r$

Each of these expressions is a factor of $p \cdot r \cdot r$, and we have listed all of them. There are 6 factors of $n = pr^2$, including 1 and itself.

Incorrect Response A: This response may have come from only counting 1, p, and r.

Incorrect Response B: This response may have come from only counting factors 1 and itself, along with p and r^2, and not counting the other possible combinations of prime factors or from counting only p, r, $p \cdot r$, and $r \cdot r$, and ignoring the factors 1 and itself.

Incorrect Response C: This response may have come from only counting factors 1 and itself, along with p, r, and r^2, and forgetting to count the factor $p \cdot r$.

It is worthwhile to note that only numbers that are perfect squares have an odd number of factors. For example, the set of factors of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}, or thought of in pairs, 1 and 36, 2 and 18, 3 and 12, 4 and 9, with 6 left by itself. For all nonsquare positive integers, the factors always come in pairs; and hence, there is an even number of factors.

13. Given $pn = 150$ where p is a prime number and n is a natural number, which of the following must be true?

    A. p is a factor of either 10 or 15.

    B. 10 is a factor of n.

    C. n is a factor of either 10 or 15.

    D. 15 is a factor of n.

Correct Response A: The question requires selecting the answer choice that *must be true* for all the ways 150 can be written as a product of p and n. To find all factorizations of this form, first find the prime factorization of 150.

$pn = 150 = 10 \cdot 15 = 2 \cdot 5 \cdot 3 \cdot 5 = 2 \cdot 3 \cdot 5 \cdot 5$

Next, list all the ways 150 can be written as a prime number times a natural number.

pn = (prime number, p)(natural number, n)
$pn = 2(3 \cdot 5 \cdot 5) = 2 \cdot 75$, or
$pn = 3(2 \cdot 5 \cdot 5) = 3 \cdot 50$, or
$pn = 5(2 \cdot 3 \cdot 5) = 5 \cdot 30$

It follows that p could be either 2, 3, or 5 and n could be either 75, 50, or 30. Now, look at each answer choice. Response A is correct because 2 is a factor of either 10 or 15 (it is a factor of 10 because it divides evenly into 10), 3 is a factor of either 10 or 15 (it is a factor of 15), and 5 is a factor of either 10 or 15 (it is a factor of both 10 and 15).

Incorrect Response B: This response is false because, while 10 is a factor of 50 and 30, 10 is not a factor of 75.

Incorrect Response C: This response is false because n is not a factor of either 10 or 15. For example, 50 is not a factor of 10 or 15. However, n is a *multiple* of either 10 or 15. That is, 75 is a multiple of 15, 50 is a multiple of 10, and 30 is a multiple of both 10 and 15. Often the concepts of multiples and factors are confused.

Incorrect Response D: This response is false because while 15 is a factor of both 30 and 75, 15 is not a factor of 50.

14. The greatest common factor of n and 540 is 36. Which of the following could be the prime factorization of n?

A. $2 \cdot 3^2$

B. $2^2 \cdot 3^3$

C. $2^4 \cdot 3^2 \cdot 7$

D. $2^4 \cdot 3^5 \cdot 5$

The greatest common factor of two positive integers is the largest number that divides both of the numbers (without a remainder). The problem states that 36 is the greatest common factor of n and 540 and asks you to determine which response *could* represent the number n. There are many possibilities for n (an infinite number, in fact), so we will go through and evaluate each response to determine which number *could* be the value of n. To do so, it is helpful to first find the prime factorization of $540 = 54 \cdot 10 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 5 = 2^2 \cdot 3^3 \cdot 5$, and $36 = 2^2 \cdot 3^2$. Working with the prime factorization of these numbers makes finding factors and dividing numbers much easier. For example, it can be seen that 36 is a factor of 540 by looking at $\frac{540}{36} = \frac{2^2 \cdot 3^3 \cdot 5}{2^2 \cdot 3^2} = 3 \cdot 5$, since you can divide common factors in the numerator and denominator to equal 1.

Now let's go through each response.

Correct Response C: Examine the number $2^4 \cdot 3^2 \cdot 7$ and divide it by the prime factorization of $36 = 2^2 \cdot 3^2$. The result is $2^2 \cdot 7$. Now, divide the prime factorization of $540 = 2^2 \cdot 3^3 \cdot 5$ by 36. The result is $3 \cdot 5$, as found before. Notice that 36 divides both numbers and that $2^2 \cdot 7$ and $3 \cdot 5$ have no common factors. Therefore, 36 is the greatest common factor of these two numbers. Since n could equal $2^4 \cdot 3^2 \cdot 7$, response C is correct.

Incorrect Response A: Notice that response A is incorrect, since 36 does not divide $2 \cdot 3^2$ without a remainder. In fact, 36 is greater than $2 \cdot 3^2 = 18$ and $\frac{18}{36} = \frac{2 \cdot 3^2}{2^2 \cdot 3^2} = \frac{1}{2}$.

Incorrect Response B: The number $2^2 \bullet 3^3 = 108$ cannot equal n because if it were n, 36 would not be the greatest common factor of 108 and 540. Instead, 108 would be the greatest common factor. This can be seen as follows. Divide $2^2 \cdot 3^3$ by the prime factorization of 36; $\frac{2^2 \cdot 3^3}{2^2 \cdot 3^2} = 3$. Next, divide 540 by 36; or $\frac{540}{36} = \frac{2^2 \cdot 3^3 \cdot 5}{2^2 \cdot 3^2} = 3 \cdot 5$. Notice that after each division, a common factor of 3 still remains. Therefore, while 36 is a common factor of both $2^2 \cdot 3^3$ and 540, it is not the *greatest* common factor of both $2^2 \cdot 3^3$ and 504. The greatest common factor of $2^2 \cdot 3^3$ and 540 is 3 times 36, or $3 \cdot (2^2 \cdot 3^2) = 2^2 \cdot 3^3 = 108$. But, we are told in the question that the greatest common factor of n and 540 is 36. Since the greatest common factor of n and 540 must be 36, n cannot equal $2^2 \cdot 3^3$. Response B is therefore incorrect.

Incorrect Response D: The number $2^4 \cdot 3^5 \cdot 5$ cannot equal *n* because if it were *n*, 36 would not be the greatest common factor of $2^4 \cdot 3^5 \cdot 5$ and 540. To see this, divide $2^4 \cdot 3^5 \cdot 5$ by $36 = 2^2 \cdot 3^2$. The result is $2^2 \cdot 3^3 \cdot 5$. Now, divide $540 = 2^2 \cdot 3^3 \cdot 5$ by $36 = 2^2 \cdot 3^2$. The result is $3 \cdot 5$. Since $2^2 \cdot 3^3 \cdot 5$ and $3 \cdot 5$ each have common factors of $3 \cdot 5$, 36 is not the greatest common factor of $2^4 \cdot 3^5 \cdot 5$ and 540, but rather $3 \cdot 5 \cdot 36 = 2^2 \cdot 3^3 \cdot 5$. Therefore, response D is incorrect. This is the same reasoning used in incorrect response B.

15. A shipping container measures 8 feet by 12 feet by 24 feet. The container is to be filled with identical cube-shaped boxes, each having sides measuring a whole number of feet. Which of the following expressions represents the smallest number of such identical boxes that could be packed into the container with no empty space remaining?

A. $\frac{8}{4} + \frac{12}{4} + \frac{24}{4}$

B. $\frac{8}{4} \bullet \frac{12}{4} \bullet \frac{24}{4}$

C. $8 \bullet 12 \bullet 24$

D. $8 + 12 + 24$

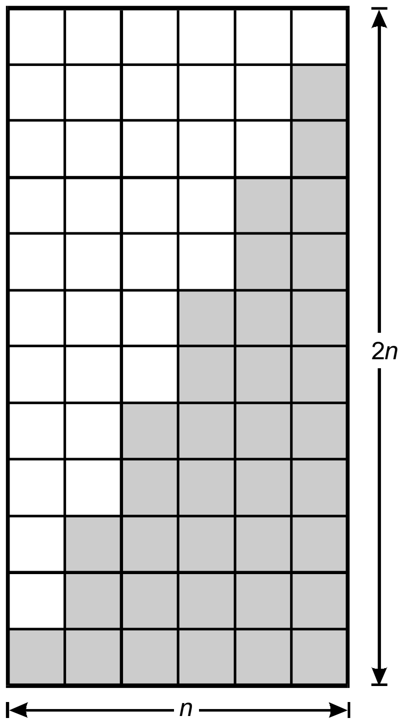
Correct Response B: The container is to be filled with identical cube-shaped boxes, and the side lengths of the cubes are whole numbers. In order to have the smallest number of cube-shaped boxes, the boxes have to be as large as possible. Since they need to pack the container so that there is no empty space, the boxes have to "fit" each of the dimensions of the shipping container. This is equivalent to saying that the side length of the boxes has to divide each dimension of the container, and it has to be the largest number that divides each dimension. Since the dimensions of the container are 8 feet by 12 feet by 24 feet, this is equivalent to finding the greatest common factor (GCF) of 8, 12, and 24. We could find the GCF using the prime factorization of the numbers, but since they are small we can find it by inspection. These numbers have common factors of 1, 2, and 4, so, 4 is the GCF. Hence, the side length of the largest box is 4 feet. To find the total number of boxes, note that the number of boxes that can fit across the 8 foot section is $\frac{8}{4} = 2$, the number that can fit across the 12 foot section is $\frac{12}{4} = 3$, and the number that can fit across the 24 foot section is $\frac{24}{4} = 6$. To find the total number of boxes requires multiplying these three numbers together. A good check of the result is to note that the volume of the container is 8 feet • 12 feet • 24 feet, or 2304 cubic feet. The volume of each cube is 4 feet • 4 feet • 4 feet or 64 cubic feet. Dividing 2,304 cubic feet by 64 cubic feet indicates that there are 36 cubes, which is the same as $2 \bullet 3 \bullet 6$ given in response B in the form $\frac{8}{4} \bullet \frac{12}{4} \bullet \frac{24}{4}$.

Incorrect Response A: This response correctly finds the number of boxes that fit each dimension, but finds their sum instead of their product. This sum is not equal to the number of boxes of side length 4 that could fill the container.

Incorrect Response C: This response results from using 1 foot as the side length of the box. This would result in the largest number of identical cubes (with whole-number side lengths) that could fill the shipping container, not the smallest number. This expression is also equal to the numerical value of the volume of the shipping container.

Incorrect Response D: This response is the sum of the dimensions of the shipping container (or the side lengths of 1 foot by 1 foot by 1 foot cube), and ignores the fact that this problem involves the volume of the container.

16. **Use the diagram below to answer the question that follows.**



The diagram above could best be used to derive a formula for which of the following quantities?

A. the sum of the first n consecutive odd integers

B. the product of the first n consecutive even integers

C. the sum of the first n consecutive even integers

D. the product of the first n consecutive odd integers

Correct Response A: Notice that in the diagram, $n = 6$ and $2n = 12$. Also notice that the number of shaded squares is one-half the total number of squares in the diagram.

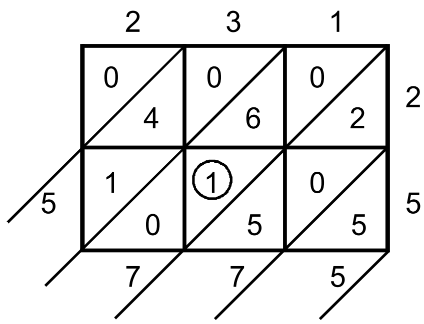
Starting at the lower left of the diagram and counting the number of shaded squares in the vertical direction gives 1, 3, 5, 7, 9, and 11. Notice that these are 6 consecutive odd integers and their sum is 36. Since half of the large rectangle is shaded, the diagram suggest that $1 + 3 + 5 + 7 + 9 + 11 = \frac{1}{2}(6)(12)$. This indicates that the sum of the first n odd integers equals $\frac{1}{2}(n)(2n) = n^2$.

Incorrect Response B: This response addresses the product of consecutive even integers, instead of the sum of consecutive odd integers.

Incorrect Response C: This response addresses the sum of consecutive even integers, instead of the sum of consecutive odd integers.

Incorrect Response D: This response addresses the product of consecutive odd integers, instead of the sum of consecutive odd integers.

17. **Use the diagram below to answer the question that follows.**



The diagram above demonstrates how the lattice multiplication algorithm is used to multiply 231 by 25 to get the product 5775. What value does the circled digit represent?

A. 1

B. 10

C. 100

D. 1000

Correct Response C: In lattice multiplication, each square in the lattice is divided along a diagonal. The place value of the two halves of the square differ by a factor of 10, the left-most half of the square having a place value of 10 times the right half. In the cell shown, the 1 digit comes from multiplying 5 times 30 to get 150. Therefore, the 1 has a value of 1×100 or 100 and the 5 has a value of 5×10 or 50.

Incorrect Response A: This response most likely comes from assuming that the value of the digit 1 is equal to the number 1.

Incorrect Response B: This response most likely comes from interpreting the values of the numbers in the square as 15 instead of as 150, resulting in a value of 10 for the digit 1.

Incorrect Response D: This response may have come from interpreting the value of the 2 and the 5 on the right as 52. This would make a product of 50 times 30, which gives 1500, resulting in a value of 1000 for the digit 1.

18. **Use the samples of a student's work below to answer the question that follows.**

$$\frac{9}{16} \div \frac{3}{4} = \frac{9 \div 3}{16 \div 4} = \frac{3}{4}$$

$$\frac{15}{8} \div \frac{5}{4} = \frac{15 \div 5}{8 \div 4} = \frac{3}{2}$$

$$\frac{5}{12} \div \frac{5}{3} = \frac{5 \div 5}{12 \div 3} = \frac{1}{4}$$

Which of the following statements best describes the mathematical validity of the algorithm that the student appears to be using?

A. It is not valid for any rational numbers.

B. It is valid only when all numerators and denominators are integers.

C. It is valid only when all numerators and denominators are positive integers.

D. It is valid for all rational numbers.

Correct Response D: Looking at the algorithm used by the student, it is clear that when dividing fractions, the student divides numerator by numerator and denominator by denominator. By performing each operation shown, it can be seen that the method works for the three examples shown. In order to verify that this method works for all rational numbers, we need to use properties of numbers and operations. A rational number is a number of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

Suppose we have two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$. The "standard" method for dividing rational numbers is as follows.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Compare the above result to $\frac{a \div c}{b \div d}$.

Since $x \div y = \frac{x}{y}$, the following steps can be taken.

$$\frac{a \div c}{b \div d} = \frac{a}{c} \div \frac{b}{d}$$

Now, apply the "standard" method for dividing rational numbers to show that the two answers will always be equal.

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{ad}{cb}$$

Notice that this is the same result obtained above, since multiplication is commutative. Furthermore, we have shown that it is true for all rational numbers.

Incorrect Response A: This response is shown to be false by the three examples given in the body of the question. All of the values given in the three examples are rational numbers, and the method works in those situations.

Incorrect Response B: This response is false. We have shown in the above discussion that the process used is valid for all rational numbers. This can be further demonstrated by creating a problem where the numerators and denominators are not integers and then working through the problem using the method given in the question and the "standard" method. For example, if two of the integers in the first example (e.g., 9 and 16) in the body of the question are replaced with nonintegers such as $\frac{5}{8}$ and $\frac{4}{5}$, and the resulting problem is simplified by first using the method given in the body of the question and then using the standard method, both methods will result in an answer of $\frac{25}{24}$.

Incorrect Response C: This response is false, as can be seen by changing the sign of any of the numbers in the problems. Changing the sign of one of the numbers will change the sign of the quotient, which will result in the correct answer.

19. **Use the inequality below to answer the question that follows.**

$3 \div x > 4 \div x$

Which of the following inequalities describes all possible values of x?

A. $x < -1$

B. $x < 0$

C. $-1 < x < 0$

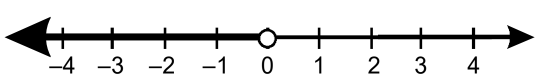
D. $0 < x < 1$

There are several ways to solve this problem. One way is to reason through the problem using number properties. The other way is to use algebraic techniques. Since the algebra for solving rational expressions (fractions with variables in the denominator) involving inequalities is subtle, it is much easier to reason through the problem by testing the numbers in each response. Notice that the question asks for the inequality that describes *all* possible values of x.

First, notice that the value of x cannot be zero, since division by zero is undefined. Next, notice that *x must be negative*, since for any positive real number, no matter how large or how small, dividing 3 by the number will always be less than dividing 4 by the same number. For example, if $x = 10$, then $3 \div 10 = 0.3$ and $4 \div 10 = 0.4$, and 0.3 is *less than* 0.4. Also, if $x = 0.1$, then $3 \div 0.1 = 30$ and $4 \div 0.1 = 40$, and 30 is *less* than 40.

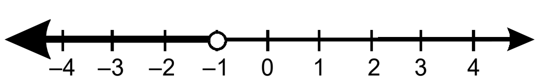
It is helpful to draw a number line for each set. Also, recall that $a > b$ if point a is located further to the right of point b on a number line. This definition is helpful when working with negative numbers.

Correct Response B: Response B describes the set of all real numbers less than 0.



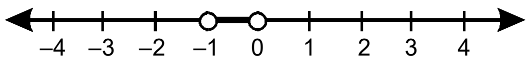
Notice that the inequality is true if x is a small, negative number close to 0. For example, suppose that $x = -0.1$. Then $3 \div -0.1 = -30$ and $4 \div -0.1 = -40$, and $-30 > -40$, since it is further to the right on the number line.

Incorrect Response A: Response A describes the set of all real numbers less than –1.



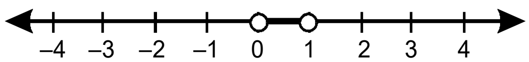
Notice that the inequality is true for the numbers less than –1. For example, if x = –2, then 3 ÷ –2 = –1.5 and 4 ÷ –2 = – 2, and it is true that –1.5 > – 2 because –1.5 is further to the right of –2 on the number line. However, this is not the correct response, since it misses the value –1 and the values between –1 and 0.

Incorrect Response C: Response C describes the set of all real numbers between –1 and 0, not including the endpoints.



The numbers in response C all satisfy the inequality, but they are not all of the values of x that satisfy the inequality. This response does not include the value –1 and all the values less than –1.

Incorrect Response D: Response D describes the set of all real numbers between 1 and 0, not including the endpoints.



None of the values in response D satisfy the inequality since they are all positive real numbers.

20. The expression ($5^{-8} \cdot 7^{-9}$) is equal to which of the following?

A. $\dfrac{1}{5(35)^8}$

B. $\dfrac{1}{7(35)^8}$

C. $\dfrac{5}{(35)^8}$

D. $\dfrac{7}{(35)^8}$

Correct Response B: From the laws of exponents, $a^{-1} = \dfrac{1}{a}$. Therefore, $(5^{-8} \cdot 7^{-9}) = \dfrac{1}{5^8 \cdot 7^9}$. The denominator of this expression contains 8 factors of 5 and 9 factors of 7. Since $7 \cdot 5 = 35$, the denominator can be rewritten to have 8 factors of 35 and one factor of 7, since there will be one extra factor of 7 remaining in the product. Using exponential notation, this is the same as $\dfrac{1}{7(35)^8}$.

Incorrect Response A: This response recognizes that there are 8 factors of 35, but interchanges the remaining factor of 7 with 5.

Incorrect Response C: This response recognizes that there are 8 factors of 35, but incorrectly places a 5 in the numerator of the expression. This expression is equal to $\dfrac{1}{5^7 7^8}$, which is incorrect.

Incorrect Response D: This response recognizes that there are 8 factors of 35, but incorrectly places a 7 in the numerator of the expression. This expression is equal to $\dfrac{1}{5^8 7^7}$, which is incorrect.

21. **Use the diagram below to answer the question that follows.**

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30
31	32	33	34	35
36	37	38	39	40

Which of the following algebraic equations could best be used to explain why, for any three-by-three cross like the one shown above, the sum of the numbers in the vertical rectangle is equal to the sum of the numbers in the horizontal rectangle?

A. $9x + 14x + 19x = 13x + 14x + 15x$

B. $5[x + (x + 1) + (x + 2)] = 5x + (5x + 5) + (5x + 10)$

C. $x + 9 + 14 + 19 = x + 13 + 14 + 15$

D. $(x – 1) + x + (x + 1) = (x – 5) + x + (x + 5)$

Correct Response D: First, observe the patterns in the table: moving left to right in any row, the numbers increase by 1; moving top to bottom in any column, the numbers increase by 5. Let x = the value of the number contained in the center cell of any three-by-three cross drawn on the table of numbers. From the pattern in the table, the element to the left of x is one less than x and has the value $x – 1$, and the element to the right is one more than x and has the value $x + 1$. Also, from the pattern in the table, the value of the cell directly above the center cell is 5 less than x, or $x – 5$, and the value directly beneath the center cell is 5 more than x, or $x + 5$. The sum of the numbers in the *horizontal* rectangle is therefore $(x – 1) + x + (x + 1)$, and the sum of the numbers in the *vertical* rectangle is $(x – 5) + x + (x + 5)$.

This problem asks for the equation of the sum of the numbers in the vertical rectangle equal to the sum of the numbers in the horizontal rectangle: $(x – 5) + x + (x + 5) = (x – 1) + x + (x + 1)$.

Response D states the equation in the opposite order of the wording of the question, $(x – 1) + x + (x + 1) = (x – 5) + x + (x + 5)$, which says that the sum of the numbers in the *horizontal* rectangle equals the sum of the numbers in the *vertical* rectangle, but the two sides of an equality can be interchanged, making the equations equivalent. Note that these two expressions simplify to $3x$, which says that each

-sum equals a value three times the middle number. In the given cross, the *vertical* and *horizontal* sums each equal 42 or 3 times the middle number, 14.

Incorrect Response A: In this response, the vertical values and the horizontal values in the given cross have been multiplied by x in an attempt to generalize. It might seem on first glance that this expression is the correct answer since both sides equal $42x$. However, this expression is only correct when $x = 1$ and is not correct anywhere else in the diagram. Consider $x = 2$ and take the top number as $9x$ or 18 in the diagram. In the diagram, the new cross has the values 18, 23, 28 vertically, not 18, 28, 38, as the expressions ($9x$, $14x$, and $19x$) would indicate. Moving from the original cross to another cross in the diagram is not multiplicative, but rather additive. In fact, all of the comparable values in the new cross in the diagram are 9 more than in the original cross. For example, 22 vs. 13; 23 vs. 14; 24 vs. 15. This pattern of adding 9 can be explored further by seeing that to get from the value 9 at the top of the original cross to the value 18 at the top of the new cross in the diagram, you could travel 2 steps down and 1 step to the left, which translates to adding 2 times 5 and subtracting 1.

Incorrect Response B: In this response the values horizontally could be expressed generally as x, $x + 1$, and $x + 2$, but multiplication by 5 results in an incorrect answer. In this answer, the right-hand side of the expression uses the distributive property to multiply 5 by each term to make the two sides equal. However, this expression will never be correct for any cross, anywhere in the diagram, since the entire right-hand side of the expression indicates three multiples of 5 going down vertically. Hence, response B is incorrect for all values of x.

Incorrect Response C: The question asks for which algebraic equation best explains why the sum of the numbers in the vertical rectangle is equal to the sum of the numbers in the horizontal rectangle. Response C is incorrect because it is not the best algebraic equation, although it does produce correct results. For any new cross that is drawn in the diagram, there is a value of x that makes the expression $x + 9 + 14 + 19$ equal the vertical sum and the expression $x + 13 + 14 + 15$ equal the horizontal sum. That value of x can be found for each new cross as follows: $x = (3)$(the middle number in the new cross minus 14). Note that 14 is the middle number in the original cross pictured in the question.

For example, make a new cross with the middle number 33. The vertical sum will be 28 + 33 + 38 = 99, and the horizontal sum will be 32 + 33 + 34 = 99. Now, find the value of the middle number of this new cross minus 14, 33 – 14 = 19. Multiplying 19 by 3 gives 57 for the value of x. Then, note that $x + 9 + 14 + 19 = 57 + 9 + 14 + 19 = 99$ for the vertical sum and that $x + 13 +14 + 15 = 57 + 13 + 14 + 15 = 99$ for the horizontal sum, therefore, the expression $x + 9 + 14 + 19 = x + 13 + 14 + 15$ is true.

The value of x will be negative if the cross is moved to a location where the middle number is less than 14. For example, make a new cross with the middle number at 8. The vertical sum of this cross will be 3 + 8 + 13 = 24, and the horizontal sum will be 7 + 8 + 9 = 24. Now, find the value of the middle number of this new cross minus 14, 8 – 14 = –6. Multiplying –6 by 3 gives –18 for the value of x. Then, note that $x + 9 + 14 + 19 = -18 + 9 + 14 + 19 = 24$ for the vertical sum and $x + 13 + 14 + 15 = -18 + 13 + 14 + 15 = 24$ for the horizontal sum, and the expression $x + 9 + 14 + 19 = x + 13 + 14 + 15$ is again true.

Also, note that the value of x for any new cross could be found by multiplying (3)(the value of any of the numbers in the new cross minus the value of the number in the comparable location in the original cross).

Response C would be true for specific vales of x (whether positive or negative) added to the sums in the original cross, but is not the "best" method of explaining why the vertical sum equals the horizontal sum, in general, which is what the question asked.

22. **Use the problem below to answer the question that follows.**

> A landscaper bought some decorative cement blocks from a landscaping supplier. The supplier charged 5% sales tax and the total came to \$315. Without the tax, the landscaper could have bought 6 more blocks for the same total cost. How many blocks did the landscaper buy?

If p represents the price of one block, in which of the following equations does x represent the answer to the problem above?

A. $0.95px = p(x + 6)$

B. $1.05p(x + 6) = 315$

C. $1.05px = p(x + 6)$

D. $0.95p(x + 6) = 315$

Correct Response C: To solve this problem, first define the variables. It is given that p = the price of one block. The problem asks for the number of blocks bought by the landscaper. Define x = the number of blocks purchased. From the problem, we know the following:

cost of x blocks + tax on the cost of x blocks = \$315
cost of x + 6 blocks = \$315

Since these two costs are equal, we can write the problem as follows:

cost of x blocks + tax on the cost of x blocks = cost of x + 6 blocks

Since p = the price of one block and x = the number of blocks bought,

px = the cost of buying x blocks.

Since there is a 5% sales tax, the tax is the cost of x blocks times 0.05, or

$0.05px$ = tax on the cost of x blocks.

Since p = the price of one block, then

$p(x + 6)$ = cost of x + 6 blocks.

Putting all of this information into the original statement of the problem results in the following:

$px + 0.05px = p(x + 6)$

Notice that the left side of this equation can be factored.

$px(1 + 0.05) = p(x + 6)$
$px(1 .05) = p(x + 6)$
$1.05px = p(x + 6)$

Incorrect Response A: This response comes from correctly determining the expressions representing the cost of x blocks, the cost of $x + 6$ blocks, and the cost of the tax, but instead of adding the tax to the cost of x blocks, the tax is added to the cost of $x + 6$ blocks as follows: $px = p(x + 6) + 0.05x$. When simplified, the tax expression ($0.05px$) is subtracted from both sides, leaving $0.95px = p(x + 6)$.

Incorrect Response B: This response results from adding the tax on the cost of $x + 6$ blocks to the cost of x + 6 blocks, and then equating that expression to $315 as follows: $p(x + 6) + 0.05p(x + 6) = 315$. Simplifying gives the result in response B. This expression is not correct, since it is the cost of $x + 6$ blocks that equals 315 dollars, not the cost plus the tax.

Incorrect Response D: This response results from finding the cost of $x + 6$ blocks and equating that cost with 315 plus the tax on the cost of $x + 6$ blocks as follows: $p(x + 6) = 315 + 0.05p(x + 6)$. Simplifying gives $0.95p(x + 6) = 315$, the equation in response D. This expression is not correct, since the cost of $x + 6$ blocks should be equal to 315 dollars.

23. A store that sells handcrafted items takes \$3.00 per item plus 40% of the sale price for each item sold. The rest of the money from item sales goes to the craftsperson. All items sold cost \$5.00 or more. If p represents the sale price of one item, which of the following expressions represents the amount of money the craftsperson gets for each item sold?

    A. $\frac{2}{5}p + 3$

    B. $\frac{2}{5}p - 3$

    C. $\frac{3}{5}p + 3$

    D. $\frac{3}{5}p - 3$

Correct Response D: We are given that p equals the sale price of an item sold. The store takes \$3 plus 40% of the price of the item. Note that 40% = 0.4. So, for example, if an item sells for \$10 the store would take \$3 + (0.4)(\$10), which in this case would be \$7. The craftsperson would be left with \$10 – (\$3 + 0.4 • \$10) or \$3. Translating this into algebra means that the store will take $3 + 0.4p$. The craftsperson will get $p - (3 + 0.4p)$. Since there is a negative sign in front of the parentheses, this can be written as $p + -1(3 + 0.4p)$ which, using the distributive property and rewriting, simplifies to $p - 3 + (-0.4p) = -3 + 0.6p = \frac{3}{5}p - 3$, since $0.6 = \frac{3}{5}$.

Incorrect Response A: This response represents the amount of money that the store collects for each craft item sold. This quantity needs to be subtracted from p to find the amount that the craftsperson will receive. Note that $0.4 = \frac{2}{5}$.

Incorrect Response B: This response most likely results from assuming that the craftsperson will get \$3 less than the 40% of the price of the item.

Incorrect Response C: This response most likely comes from failing to correctly simplify an expression with a negative sign in front of the parentheses, which is a very common algebraic error. It is essential that a negative sign in front of parentheses be distributed throughout the terms inside the parentheses.

24. **Use the solution procedure below to answer the question that follows.**

$-3x + 25 = 4$
$-3x + 25 - 25 = 4 - 25 = -21 \div (-3) = 7$
$x = 7$

Which of the following is a major flaw in the procedure shown above?

A. The concept of the opposite of a number is confused with subtraction.

B. The equal sign is used to connect expressions that are not equal.

C. The solution contains an error in the arithmetic of signed numbers.

D. The order of operations between subtraction and division is reversed.

Correct Response B: In analyzing the steps in the solution and providing a justification for each step, we begin with the equation below, which represents the information given in the question.

$-3x + 25 = 4$

The next step uses the subtraction property of equality, which says that you can subtract the same number from both sides of an equation and both sides of the equation will still be equal. This step was properly carried out, as shown below.

$-3x + 25 - 25 = 4 - 25$

Notice that an error is introduced in the next step of the equation.

$-3x + 25 - 25 = 4 - 25 = -21 \div (-3)$

While $4 - 25 = -21$, it does not equal $-21 \div (-3)$, since $-21 \neq 7$. Therefore, response B is the correct response. The equal sign is used to connect expressions that are not equal.

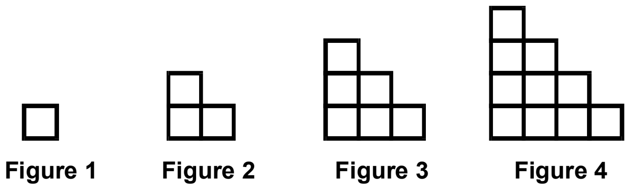
It is important to point out that while the solution to the equation is correct, as can be verified by substituting $x = 7$ into the original equation, an error is made in the problem-solving process, since the equal sign was used incorrectly. Misuse of the equal sign indicates a misconception about the nature of equality and can lead to student confusion and errors.

Incorrect Response A: No error was made in confusing the opposite of a number with subtraction. While 25 was subtracted from both sides of the equation, this is equivalent to adding the opposite of 25, or -25, to both sides of the equation, since subtraction is equivalent to adding the opposite.

Incorrect Response C: No error was made involving the arithmetic of signed numbers since $4 - 25 = -21$ and $-21 \div (-3) = 7$.

Incorrect Response D: No error was made involving the order of operations. The most efficient way to solve equations of this kind is to first transform the equation by using the addition (or subtraction) property, and then using the multiplication (or division) property.

25. **Use the diagram below to answer the question that follows.**



If the pattern continues, how many more small squares are in figure 100 than are in figure 99?

A. 98

B. 99

C. 100

D. 101

Correct Response C: Carefully analyze the pattern to determine how the figure is changed in going from one figure to the next figure.

To get figure 2, 2 squares are added to figure 1.
To get figure 3, 3 squares are added to figure 2.
To get figure 4, 4 squares are added to figure 3.

This can be visualized as adding another set of small squares that run along the stepped diagonal of the figure. From this pattern, the following rule emerges.

$n + 1$ squares are added to figure n to get figure $n + 1$.

In other words, 100 squares are added to figure 99 to get figure 100. Therefore, figure 100 has 100 more squares than figure 99.

Incorrect Response A: This response might have come from analyzing the pattern incorrectly in the following way: 1 square is added to figure 2 to get figure 3, 2 squares are added to figure 3 to get figure 4, etc. In other words, $n - 1$ squares are added to figure n to get figure $n + 1$. Using this rationale, 98 squares would be added to figure 99 to get figure 100.

Incorrect Response B: This response might have come from the following incorrect analysis. One square is added to figure 1 to get figure 2, 2 squares are added to figure 2 to get figure 3, 3 squares are added to figure 3 to get figure 4, etc. In other words, n squares are added to figure n to get figure $n + 1$. Using this rationale, 99 squares are added to figure 99 to get figure 100.

Incorrect Response D: This response may have come from analyzing the pattern incorrectly in the following way. Three squares are added to figure 1 to get figure 2, 4 squares are added to figure 2 to get figure 3, 5 squares are added to figure 3 to get figure 4, etc. In other words, $n + 2$ squares are added to figure n to get figure $n + 1$. Therefore, 101 squares would be added to figure 99 to get figure 100.

26. **Use the table below to answer the question that follows.**

y		$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
	4	20	28	36	44	52
	3	15	23	31	39	47
	2	10	18	26	34	42
	1	5	13	21	29	37
	0	0	8	16	24	32
		0	**1**	**2**	**3**	**4**

Each number in the table above represents a value of w that is determined by the values of integers x and y. For example, when $x = 2$ and $y = 1$, $w = 21$. If the pattern continues, what is the value of w when $x = 20$ and $y = 8$?

A. 164

B. 200

C. 208

D. 820

Correct Response B: From the information given in the table, the value of w is the entry in the table for a given column (x) and row (y). When $x = 2$ and $y = 1$, $w = 21$, which corresponds to the entry in the table of the column labeled 2 and the row labeled 1. We are asked to assume that the pattern in the table continues and to find the value of w when $x = 20$ and $y = 8$. This requires analyzing the pattern in the table and extending the pattern in order to find the element in the table for the row labeled $y = 8$ and the column labeled $x = 20$.

One way to solve this problem is to notice that the entries in each row increase by 8 (from left to right) and that the entries in each column increase by 5 (from bottom to top). Since the table has the value $w = 0$ when $x = 0$ and $y = 0$, the value of any element in the table can be found by multiplying the x value by 8 and the y value by 5 and then adding the two numbers. For the values in the question, $x = 20$ and $y = 8$, this can be expressed as $w = 8(20) + 5(8) = 200$. This can also be written algebraically as $w = 8x + 5y$.

Another way to see this is to start in the $x = 0$ column and extend the pattern up the column through to $y = 8$ to get $w = 8(5) = 40$. Then, still in the $y = 8$ row, move over 20 columns while following the pattern by adding $8(20) = 160$ to 40 to get $w = 200$ for the entry when $x = 20$ and $y = 8$.

A more detailed approach is to first look at row zero (i.e., the set of all *x*'s corresponding to $y = 0$). Notice that the value of each element increases by 8 when moving from the left to the right. Looking at the rest of the rows ($y = 1$, $y = 2$, etc.) shows that this pattern holds for all of the rows in the table.

Next, look at the columns of the table. Start with column zero (i.e., the set of all *y*'s corresponding to $x = 0$). Notice that the value of each element increases by 5 when going from the bottom to the top. Analyzing the rest of the columns ($x = 1$, $x = 2$, etc.) shows that the pattern holds for all of the columns in the table.

Now extend the pattern. Starting with row $y = 0$, note that the value in the row is given by the product of 8 and *x*. Therefore, starting with row $y = 0$ at the bottom of the column labeled $x = 20$, the elements, listed from the bottom to the top, are 160, 165, 170, 175, 180, 185, 190, 195, 200, 205, etc. This means that if $x = 20$ and $y = 0$, $w = 160$. If $x = 20$ and $y = 1$, $w = 165$, if $x = 20$ and $y = 2$, $w = 170$, etc. Continuing in this way, if $x = 20$ and $y = 8$, then $w = 200$.

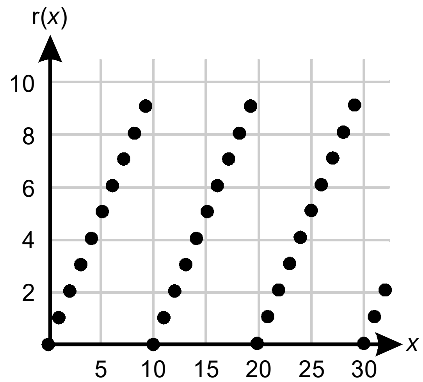
Incorrect Response A: This response might have resulted from correctly extending the pattern in the table, but mistakenly interchanging the rows and the columns in the table. The question asks for the value of *w* when $x = 20$ and $y = 8$. However, this response gives the value of *w* when $x = 8$ and $y = 20$. This can be seen from similar reasoning to the above. To show this, this time we will extend the table by starting with the column $y = 20$. Notice that when $y = 20$ and $x = 0$, *w* = the product of 5 and 20 or $w = 100$. Next, in moving from left to right, each value increases by 8. Therefore, moving from $x = 0$ to $x = 8$ is the same as adding 64 to 100. The answer is then $w = 164$.

Incorrect Response C: This response might have come from guessing a pattern using only the numbers given in the body of the problem, and not analyzing the pattern in the table. It is given in the question that when $x = 2$ and $y = 1$, $w = 21$. This response assumes that the value of *w* is found by combining the digits 2 and 1 to get $w = 21$. The same approach is used with $x = 20$ and $y = 8$. Combining these digits gives $w = 208$.

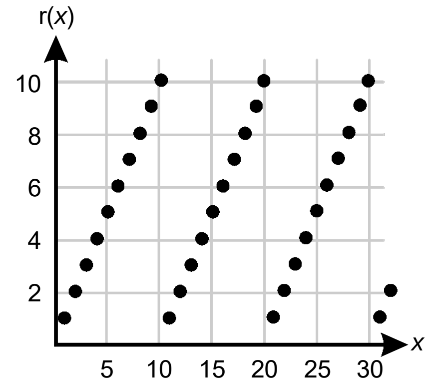
Incorrect Response D: This response might have come from guessing a pattern using only the numbers given in the body of the problem, and not analyzing the pattern in the table. A further mistake is introduced by mixing up the order of the digits. It is given in the question that when $x = 2$ and $y = 1$, $w = 21$. This value is found by combining the digits 2 and 1 to get 21. A similar approach is used with $x = 20$ and $y = 8$, but an additional error is made by transposing the digits and reading the information as $x = 8$ and $y = 20$ to get 820.

27. The function r(x) gives the remainder when a whole number x is divided by 10. Which of the following graphs represents r(x)?

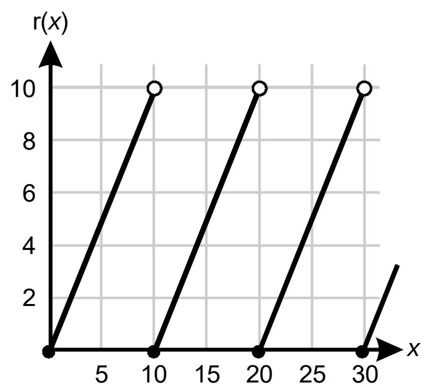
A.



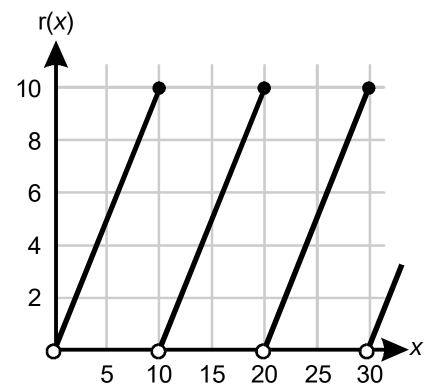
B.



C.



D.



Correct Response A: This question combines the ideas of functions and number theory. First, it is important to determine what the graph in the response is saying. The graph displays the remainder, r(x), of dividing a whole number, x, by 10. For example, if x = 12, then r(12) = 2, since the remainder of dividing 12 by 10 is 2. Therefore, the ordered pair (x, y) with y = r(x) corresponding to x = 12 is (12, 2). Note that the horizontal scale on the graph is different from the vertical scale.

If a number less than 10 is divided by 10, it can still have a nonzero remainder. This can be seen by looking at division of integers. 12 divided by 10 is equal to 1 with a remainder of 2 because 12 = 10(1) + 2. In a similar manner, 1 divided by 10 is 0 with a remainder of 1, since 1 = 10(0) + 1. 2 divided by 10 equals zero with a remainder of 2, since 2 = 10(0) + 2.

This way of thinking about division of integers is based on the division algorithm. Note that this is different from the long division algorithm. A more detailed discussion of the division algorithm is given at the end of this analysis.

We can now determine some values of the remainder of x divided by 10.

Several examples of r(x) are shown below for whole numbers less than 10.

1 = 10(0) + 1, so for x = 1, r(1) = 1 (r = 1 when 1 is divided by 10)
2 = 10(0) + 2, so for x = 2, r(2) = 2 (r = 2 when 2 is divided by 10)
3 = 10(0) + 3, so for x = 3, r(3) = 3 (r = 3 when 3 is divided by 10)

More examples are given below for numbers greater than or equal to 10.

10 = 10(1) + 0, so for x = 10, r(10) = 0 (r = 0 when 10 is divided by 10)
11 = 10(1) + 1, so for x = 11, r(11) = 1 (r = 1 when 11 is divided by 10)
15 = 10(1) + 5, so for x = 11, r(15) = 5 (r = 5 when 15 is divided by 10)
20 = 10(2) + 0, so for x = 20, r(20) = 0 (r = 0 when 20 is divided by 10)

These are the values that are shown in the graph in response A. The values shown are closed dots, which are used to indicate that the values are discrete whole numbers and not a continuous set of numbers. In fact, the graph of r(x) consists of only a discrete set of whole numbers. Continuous sets that have numbers between the values are indicated using a solid line, as in incorrect responses C and D.

Incorrect Response B: In response B, notice that the graph says that the remainder of 10 divided by 10 equals 10, which is not correct. The graph also indicates that the remainder of 20 divided by 10 is 10, and that the remainder of 30 divided by 10 is 10, neither of which is correct.

Incorrect Response C: This graph is incorrect since it indicates that the graph is continuous. The graph cannot be continuous because the remainder will always be a whole number less than 10. Notice, however, that this graph does give the correct value for the remainder of 10 divided by 10 [i.e., r(10) = 0], the remainder of 20 divided by 10 [i.e., r(20) = 0], and the remainder of 30 divided by 10 [i.e., r(30) = 0], since there is a solid dot on the graph corresponding to x = 10, r(10) = 0, etc.

Incorrect Response D: This graph is incorrect since it indicates that the graph is continuous. The graph cannot be continuous because the remainder will always be a whole number less than 10. Also note that the graph gives the incorrect values for the remainder of 10 divided by 10 [i.e., r(10) = 10], the remainder of 20 divided by 10 [i.e., r(20) = 10] and the remainder of 30 divided by 10 [i.e., r(30) = 10], since there are closed dots on (10, 10), (20, 10), and (30, 10). Also, there are open dots on (10, 0), (20,0), and (30, 0), which indicate that these points are not included in the graph.

Note: Division of whole numbers is based on the division algorithm. The division algorithm says that for any two whole numbers a and b, with $b \neq 0$, there exist numbers q and r such that $a = qb + r$, where $0 \leq r < b$. Here q is called the *quotient* and r is called the *remainder*. For example, if a = 10 and b = 3, we can write that 10 = 3(3) + 1. Hence, q = 3 and r = 1. However, the theorem says that this algorithm works for *any* two numbers, so we could also reverse a and b to get a = 3 and b = 10. This results in 3 = 10(0) + 3. Here q = 0 and the remainder r = 3. Hence, if we want to determine r(x), the remainder of a number x when divided by 10, we can use the division algorithm.

28. Four pumps begin draining a 5400-gallon pool. At the same time, two pumps begin draining a 4000-gallon pool. Assuming that all of the pumps drain at the same rate, how many gallons are left in the smaller pool when the larger pool is finished being drained?

A. 1300 gallons

B. 1350 gallons

C. 2700 gallons

D. 2750 gallons

Correct Response A: It is given that 4 identical pumps begin draining a 5400-gallon pool. At the same time, 2 of the same pumps are draining a 4000-gallon pool, and the pumps all start draining at the same time. The question asks to determine how many gallons are left in the smaller pool after the larger pool has been drained.

One way to solve this problem is to reason as follows. If 4 pumps can drain 5400 gallons of water in a given amount of time, then 2 pumps should be able to drain one-half of 5400 gallons in the same amount of time. Therefore, there should be $\frac{5400}{2}$ = 2700 gallons drained from the 4000-gallon pool in the same amount of time. There will then be 4000 gallons – 2700 gallons = 1300 gallons remaining in the 4000-gallon pool.

Since it is rather convenient that the numbers work well in this problem, but might not in another problem, it would be useful to solve the problem with a more general approach through the use of algebra.

We need to determine how long it will take to drain the larger pool. It will then be possible to determine how many gallons were pumped out of the smaller pool in the same amount of time.

The rate at which a pump pumps is given by the number of gallons pumped per unit time. The problem does not give the rate at which the pumps pump water, so we will let R = rate at which one pump pumps water. We can then write the following *rate equation*.

$$R = \frac{\text{number of gallons}}{\text{time}} = \frac{N}{t} \text{ (equation 1)}$$

where N = the number of gallons being pumped and t = the amount of time for one pump to pump the gallons of water.

This can be expressed in terms of t, the amount of time it takes for one pump to empty the pool, as follows.

Start with the rate equation (equation 1) and put it in terms of *t* by multiplying both sides by *t* and then dividing both sides by *R*:

$R = \frac{N}{t}$ (equation 1)

$tR = N$

$t = \frac{N}{R}$

This equation represents the amount of time *t* required for one pump to pump *N* gallons of water at a rate *R*.

For the larger pool $N = 5400$ and there are four pumps. Therefore, the rate at which 4 pumps can empty the pool is 4 times the rate at which one pump can empty the pool. The amount of time it takes for 4 pumps to empty the 5400 gallon pool is given by the following.

$t = \frac{N}{4R} = \frac{5400}{4R}$ (equation 2)

We now need to determine how many gallons (*N*) were pumped from the 4000-gallon pool in the same length of time. Start with the rate equation (equation 1) and solve it for *N*.

$R = \frac{N}{t}$ (equation 1)

$\frac{N}{t} = R$

Multiply both sides by *t* to get the following.

$N = Rt$

Here, *R* is the rate that one pump pumps water. Since two pumps are used to empty the pool, the rate for the 4000-gallon pool is given by $2R$. Therefore, the number of gallons that two pumps can pump in time *t* is given by the following.

$N = 2Rt$ (equation 3)

We can now use *t*, the amount of time it takes for 4 pumps to empty the 5400 gallon pool. This is given by equation 2. Substitution of equation 2 into equation 3 gives

$N = 2Rt = 2R \bullet \frac{5400}{4R} = \frac{5400}{2} = 2700$ gallons.

This is the same answer we found above.

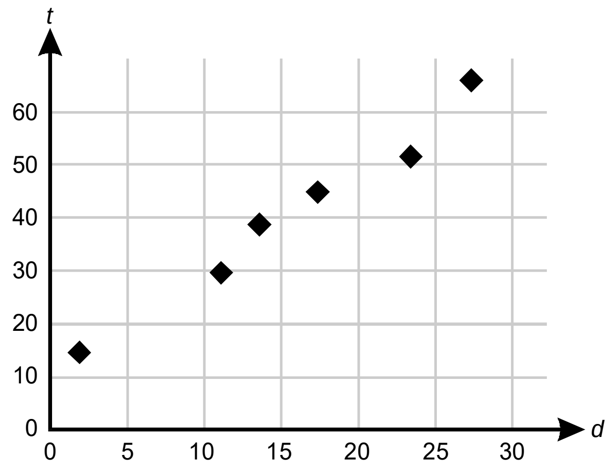
Since the question asks for how many gallons remain in the 4000 gallon pool, the answer is given by 4000 gallons – 2700 gallons = 1300 gallons.

Incorrect Response B: This response might have come from reasoning that the four pumps emptying the 5400-gallon pool each pump one-fourth of 5400 gallons or 1350 gallons, and concluding that this is how many gallons will remain.

Incorrect Response C: This response represents the number of gallons that were pumped from the 4000-gallon pool, not the number of gallons that remain in the pool.

Incorrect Response D: This response might have come from reasoning that the four pumps emptying the 5400-gallon pool each pump one-fourth of 5400 gallons or 1350 gallons, forgetting that there were 2 pumps draining the 4000-gallon tank, and making a further error when subtracting 4000 – 1350 to get 2750 gallons instead of 2650 gallons.

29. **Use the graph below to answer the question that follows.**



The graph above shows the distance *d* in miles and the time *t* in minutes for six bus routes around a city. Which of the following equations best models the relationship between *d* and *t* for these bus routes?

A. $t = d$

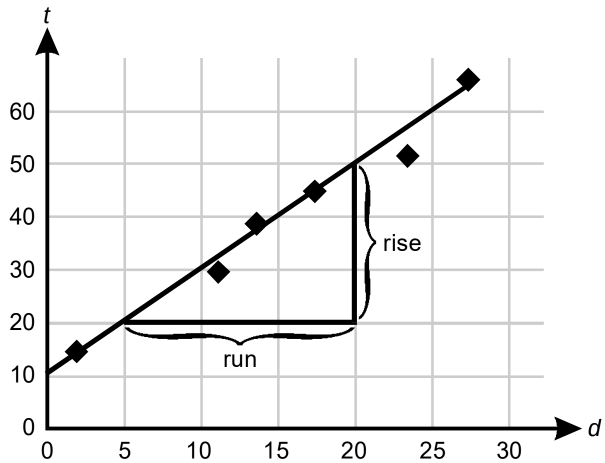
B. $t = d + 10$

C. $t = 2d$

D. $t = 2d + 10$

Correct Response D: The graph shows several points on a rectangular coordinate system. The question asks for the equation that best models the relationship between distance *d* and time *t*. Notice that the points, for the most part, seem to all lie on a line. Although there is some variation from a line, it is reasonable that a line will be a very good representation of the relationship. This relationship can be found by drawing a line that seems to fit the data points and then finding the equation of that line.

An example of such a line is shown below. This line is produced by simply drawing a line so that all the data points are near the line. While there are many such lines that could be drawn, most of them will be fairly good representations, and we can use the answer choices to see which is closest.



Note that the scales on the horizontal axis and vertical axis are different in the given graph. The horizontal axis has a scale of 5 units and the vertical axis has a scale of 10 units.

One way to find the equation of this line uses the slope-intercept form of the equation of a line, $y = mx + b$, where m = the slope of the line and b = the y-intercept of the line. In this case, y corresponds to time t and x corresponds to distance d. From the graph, the y-intercept is close to (0, 10), so let $b = 10$. The slope of the line is the rise over the run. To determine the slope, a triangle is drawn on the graph indicating the rise and the run.

$$m = \frac{\text{rise}}{\text{run}} = \frac{30}{15} = 2$$

Using the slope-intercept form of a linear equation, $y = mx + b$, this gives $y = 2x + 10$. In terms of the variables t and d on the graph, the equation is

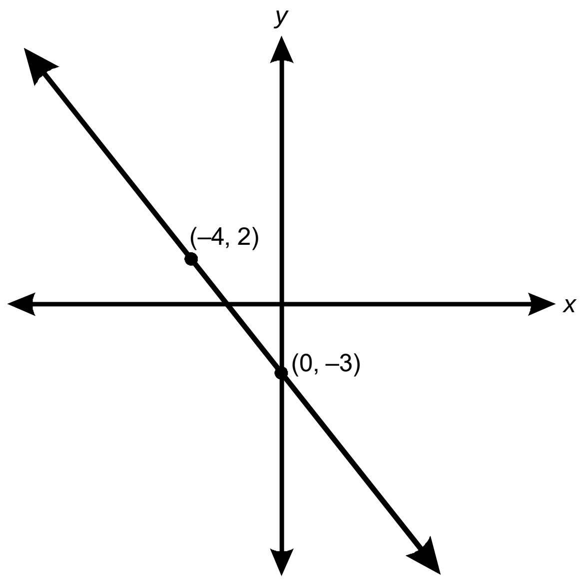
$$t = 2d + 10.$$

Incorrect Response A: This answer represents a line with a slope of 1 that passes through the origin of the coordinate system.

Incorrect Response B: This answer has the correct vertical intercept of the line, namely t = (0, 10), but it has a slope of 1. This answer for the slope might have resulted from not noticing the different scales on the axes of the graph. If the scales were the same, the slope would be 1.

Incorrect Response C: This answer comes from correctly finding the slope of the line that passes through the origin of the coordinate system but not accounting for the y-intercept, (0, 10).

30. **Use the graph below to answer the question that follows.**



The graph above represents the equation $Wx + 4y = -12$.
What is the value of W?

A. –6

B. –3

C. 3

D. 5

Correct Response D: Given an equation of a line, if the coordinates of any point on that line are substituted into the equation, those coordinates will make that equation a true statement. Note that (–4, 2) means that $x = -4$ and $y = 2$.

Substituting the coordinates of the point (–4, 2) into the equation $Wx + 4y = -12$ gives the following.

$$W(-4) + 4(2) = -12$$

Then solve for W.

$$W(-4) + 8 = -12$$
$$-4W + 8 - 8 = -12 - 8$$
$$-4\,W = -12 - 8$$
$$-4W = -20$$
$$-4W \div (-4) = -20 \div (-4)$$
$$W = 5$$

Note that if you had used the coordinates of the point (0, –3), you would get $W(0) + 4(-3) = -12$, but the W-term would be eliminated. You still end up with a true statement ($-12 = -12$), but you can't solve for W.

Another way to solve this problem is to find the slope-intercept form of the equation of a line, and then transform the equation to the form given in the question, which is called the standard form of the equation of a line. This is a more general method for finding the equation of a line given points. In this problem, it is much less efficient than the previous solution. We will quickly go through the method.

$$y = mx + b$$

Find the slope, considering the first point as (x_1, y_1) and the second point as (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 2}{0 - (-4)} = \frac{-5}{4}$$

Rewrite the equation using the slope-intercept form ($y = mx + b$) with the value of the slope m found above.

$$y = \frac{-5}{4}x + b$$

Substitute the coordinates of any point into the equation to find b. We will use the point (0, –3), so $x = 0$ and $y = -3$.

$$-3 = \frac{-5}{4}(0) + b$$
$$b = -3$$

Notice that the point (0, –3) is the point where the line crosses the y-axis, so the y-coordinate at the y-intercept of the line, $b = -3$, could also have been found from inspecting the graph.

The equation of the line in slope-intercept form is now known.

$$y = \frac{-5}{4}x + (-3) \text{ or } y = \frac{-5}{4}x - 3$$

Transform the equation by multiplying both sides by 4.

$$4y = -5x - 12$$

Add $5x$ to both sides.

$$5x + 4y = -12$$

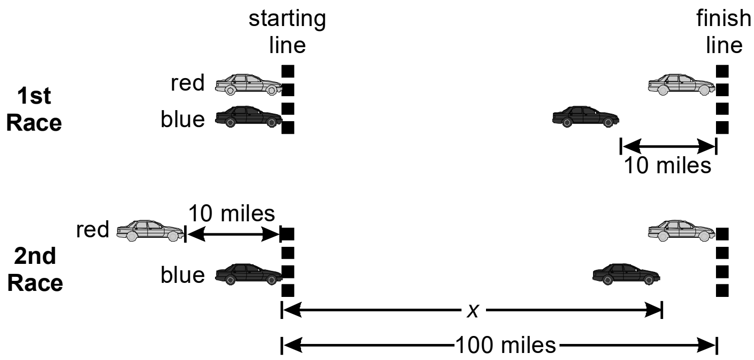
Since W is the coefficient of x, $W = 5$.

Incorrect Response A: This answer might have come from using the formula for slope and getting 4 in the denominator but incorrectly getting 6 in the numerator by multiplying –3 and –2 instead of adding them. From the graph, the *y*-intercept is (0, –3), so the equation is then incorrectly written as $y = \frac{6}{4}x + (-3)$. Transforming the equation results in $-6x + 4y = -12$ and $W = -6$.

Incorrect Response B: This response might have come from confusing the value of *W*, which is the coefficient of *x* in the given equation, with the *y*-intercept of the line passing through the point (0, –3).

Incorrect Response C: This response might have come from mixing the coordinates of the two points on the line and substituting $x = -4$ and $y = 0$ into the equation $Wx + 4y = -12$. This would give $(-4)(W) + 4(0) = -12$ or $-4W = -12$ and $W = 3$.

31. **Use the problem below to answer the question that follows.**



A red car and a blue car compete in two 100-mile races. In the first race, both cars leave the starting line at the same time. When the red car crosses the finish line, the blue car has 10 miles left to go. In the second race, both cars start at the same time, but while the blue car begins at the starting line, the red car begins 10 miles behind the starting line.

Assuming that each car's average speed does not change, how far has the blue car traveled in the second race when the red car reaches the finish line?

Which of the following proportions could be used to solve the problem above?

A. $\frac{100}{90} = \frac{110}{x}$

B. $\frac{100}{10} = \frac{110}{x}$

C. $\frac{100}{90} = \frac{90}{x}$

D. $\frac{100}{10} = \frac{90}{x}$

Correct Response A: The average speed of an object (*s*) is given by the distance traveled (*d*) divided by the time of travel (*t*). For example, in the United States we measure speed in miles per hour, which is distance (miles) divided by (per) time (hours).

One way to solve this problem is to reason as follows. Average speed equals distance divided by time. The problem states that the average speeds of the cars are the same during each race. Also, the times traveled for each car are the same per race. It then follows that the average speed for each race is proportional to the distance traveled by each car. Therefore, the following ratios must hold, where *d* represents the distance traveled.

$$\frac{d_{redrace1}}{d_{bluerace1}} = \frac{d_{redrace2}}{d_{bluerace2}}$$

Putting the appropriate distances into the equation above gives the following result.

$$\frac{100}{90} = \frac{110}{x}$$

Since the reasoning given above is subtle, it is instructive to solve this problem using techniques of algebra. Note that average speed (s) is given by the following equation:

$$s = \frac{\text{distance}}{\text{time}} = \frac{d}{t}$$

For the first race, the average speed of the red car is given by the distance traveled (100 miles) divided by the time of travel (t_1).

$$s_{redrace1} = \frac{\text{distance}}{\text{time}} = \frac{d}{t} = \frac{100 \text{ miles}}{t_1}$$

The average speed of the blue car for the first race is given by the distance traveled divided by the time of travel (t_1). From the diagram (not drawn to scale) the distance traveled is (100 miles – 10 miles). The time (t_1) is the same for each car, since they both travel for the same length of time. The blue car travels 90 miles in the same amount of time that the red car travels 100 miles.

$$s_{bluerace1} = \frac{\text{distance}}{\text{time}} = \frac{d}{t} = \frac{90 \text{ miles}}{t_1}$$

The average speed of the red car for the second race is given by the distance traveled (100 miles + 10 miles) divided by the time of travel (t_2) for the second race.

$$s_{redrace2} = \frac{\text{distance}}{\text{time}} = \frac{d}{t} = \frac{110 \text{ miles}}{t_2}$$

The average speed of the blue car for the second race is given by the distance traveled (x) divided by the time of travel (t_2). The time of travel for the second race (t_2) is the same for each car, since they both travel for the same length of time in the second race. Notice, however, that t_1 cannot equal t_2. In fact, $t_2 > t_1$ since, for example, it will take the red car longer to drive 110 miles at the same speed in the second race than it will to travel the 100 miles in the first race.

$$s_{bluerace2} = \frac{\text{distance}}{\text{time}} = \frac{d}{t} = \frac{x}{t_2}$$

There are now several ways to proceed to find an equation for x. They are all based on the fact, given in the problem, that the average speed of the red car is the same for both races and the average speed of the blue car is the same for both races.

One way to solve these equations is to set up a system of two equations, and solve the system using algebra. Since the average speeds of each car during the races are the same, the speed of the red car in race 1 is the same as its speed in race 2.

$s_{redrace1} = s_{redrace2}$ (fact 1)

Using the expressions found above for the red car results in the following equation.

$$\frac{100 \text{ miles}}{t_1} = \frac{110 \text{ miles}}{t_2}$$

Solve this equation for t_1 in terms of t_2.

$$t_2 \bullet 100 = t_1 \bullet 110$$

$$t_1 = \frac{100t_2}{110} \text{ (equation 1)}$$

The speed of the blue car is the same in each race.

$s_{bluerace1} = s_{bluerace2}$ (fact 2)

Using the expressions found above for the blue car results in the following equation.

$$\frac{90 \text{ miles}}{t_1} = \frac{x}{t_2}$$

Solve this equation to get t_1.

$$t_2 \bullet 90 = t_1 \bullet x$$

$$t_1 = \frac{90t_2}{x} \text{ (equation 2)}$$

Setting equation 1 equal to equation 2 gives the following.

$\frac{100t_2}{110} = \frac{90t_2}{x}$, which is equivalent to $\frac{100}{110} = \frac{90}{x}$.

This equation can be transformed by multiplying both sides by 110 and dividing both sides by 90 to get $\frac{100}{90} = \frac{110}{x}$, the equation in response A.

Another way to solve the problem is to recognize that since the average speeds are the same, then their ratios must be the same.

$$\frac{s_{redrace1}}{s_{bluerace1}} = \frac{s_{redrace2}}{s_{bluerace2}}$$

$$\frac{\frac{100}{t_1}}{\frac{90}{t_1}} = \frac{\frac{110}{t_2}}{\frac{x}{t_2}}$$

This complex fraction can be simplified in two ways. One way is to notice that in each denominator of the complex fractions the values for t_1 on the left and t_2 on the right can be divided to give $\frac{100}{90} = \frac{110}{x}$.

The complex fraction can also be simplified algebraically as follows. Note that the top fraction is divided by the bottom fraction and apply the rule for division of fractions (invert and multiply). The left side of the equation becomes

$$\frac{\frac{100}{t_1}}{\frac{90}{t_1}} = \frac{100}{t_1} \div \frac{90}{t_1} = \frac{100}{t_1} \bullet \frac{t_1}{90} = \frac{100}{90}\text{, since } \frac{t_1}{t_1} = 1.$$

The right side of the equation becomes

$$\frac{\frac{110}{t_2}}{\frac{x}{t_2}} = \frac{110}{t_2} \div \frac{x}{t_2} = \frac{110}{t_2} \bullet \frac{t_2}{x} = \frac{110}{x}\text{, since } \frac{t_2}{t_2} = 1.$$

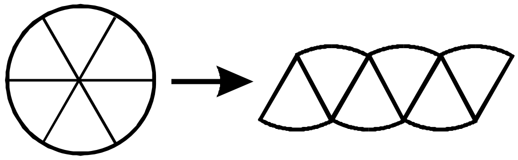
Setting the left side and right side equal gives the answer $\frac{100}{90} = \frac{110}{x}$.

Incorrect Response B: This response might have come from assuming that the ratio of the distance traveled by the red car in race 1 to the extra distance it traveled in race 2 is equal to the ratio of the distance it traveled in race 2 to the distance the blue car traveled in race 2.

Incorrect Response C: This response might have come from assuming that the ratio of the distance traveled by the red car in race 1 to the distance traveled by the blue car in race 1 is equal to the ratio of the distance traveled by the blue car in race 1 to the distance traveled by the blue car in race 2.

Incorrect Response D: This response may have come from assuming that the ratio of the distance traveled by the red car in race 1 to the increase in the distance traveled by the red car in race 2 is equal to the ratio of the distance traveled by the blue car in race 1 to the distance traveled by the blue car in race 2.

32. **Use the diagram below to answer the question that follows.**



The diagram above is used to describe the relationship between the circumference c, the radius r, and the area A of a circle. Assuming that the circle is divided into enough sections so that the figure on the right approximates a rectangle, which of the following relationships is demonstrated?

A. $A = \frac{1}{2}cr$

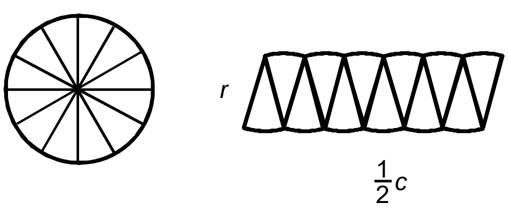
B. $A = cr$

C. $A = \frac{3}{2}cr$

D. $A = 2cr$

Correct Response A: The diagram shows a circle of radius r and circumference c. The circle is sliced into 6 sectors, and the sectors are rearranged to form a figure that resembles a parallelogram with a curved base. The area of each sector is the same. The entire circle is represented in the rearranged figure, so the area of the circle equals the area of the rearranged figure. Notice that the top of the figure is half the circumference (6 of 12 arcs), as is the bottom base. Also notice that the height of the figure is equal to the radius of the circle.

The question states that the circle is divided into enough sectors so the rearranged figure approximates a rectangle. The diagram below shows a circle that is sliced into 12 sectors.



Imagine if the circle were next sliced into 24 sectors and rearranged, then 48 sectors and rearranged, then 96 sectors and rearranged, etc. Notice that as the circle is divided into smaller and smaller sectors, the shape formed when the sectors are reassembled becomes closer and closer to a rectangle. In this case, the straight side of the figure becomes more vertical and closely represents the width of a rectangle, while the base becomes less curved and closely represents the length of a rectangle. No matter how many sectors the circle is divided into, the area of the circle equals the area of the rectangular figure. The width of the rectangle is the radius of the circle or r, and the base of the rectangle is equal to one-half the circumference of the circle or $\frac{1}{2}c$. Since the area of a rectangle is equal to the length times width, the area of the circle is equal to r times $\frac{1}{2}c$. Therefore, $A = \frac{1}{2}cr$.

Incorrect Response B: This answer comes from interpreting the base of the figure as having length equal to the circumference instead of one-half the circumference.

Incorrect Response C: This answer might have come from applying the formula for the area of a triangle to each circular sector of altitude r (in the original diagram showing 6 sectors), but interpreting the base as $\frac{1}{2}c$ instead of $\frac{1}{6}c$, and then multiplying the result by 6. Since the area of a triangle is $\frac{1}{2}$ × altitude × base, this approach would give $A_{\text{triangle}} = \left(\frac{1}{2}r\right)\left(\frac{1}{2}c\right) = \frac{1}{4}cr$ for a single triangle. Multiplying this area of a single triangle by 6 would result in $A = \left(\frac{6}{1}\right)\left(\frac{1}{4}cr\right) = \frac{3}{2}cr$.

Incorrect Response D: This answer might have come from interpreting the base of the figure as having length equal to twice the circumference instead of one-half the circumference to give $2cr$ or from thinking that the two bases each are $\frac{1}{2}c$ and the two heights are r and attempting to find the perimeter using c and $2r$, but multiplying instead of adding to give $2cr$.

33. A pretzel company sells pretzels in a cylindrical container with a radius of 10 cm and a height of 30 cm. The company's packaging designers are considering switching to a new cylindrical container with a radius of 20 cm and a height of 15 cm. How does the volume of the proposed new container compare to the volume of the old container?

A. The volume of the new container is 125 cm^3 less than the volume of the old container.

B. The volume of the new container is 5 cm^3 less than the volume of the old container.

C. The volume of the new container is equal to the volume of the old container.

D. The volume of the new container is twice the volume of the old container.

Correct Response D: One way to answer this question is to determine how the volume of a cylinder depends on its radius and its height. From the numbers given in the problem, the company is going to double the radius of the base of the cylinder (from 10 cm to 20 cm), and reduce the height of the cylinder by one-half (from 30 cm to 15 cm). Let $V_1 = \pi r^2 h$ be the volume of the original cylinder, where r is the radius and h is the height. Doubling the radius and reducing the height by one-half will result in a volume for the new cylinder (given by V_2) to be $V_2 = \pi(2r)^2\left(\frac{1}{2}h\right) = \pi 4r^2\left(\frac{1}{2}h\right) = 2\pi r^2 h = 2V_1$. This shows that the volume of the new container is twice the volume of the old container.

Another way to answer this question is to use the values given, compute the volumes, and then compare the two volumes.

Since the height of the original cylinder h_1 = 30 cm and the radius r_1 = 10 cm, the volume of the original cylinder is $V_1 = \pi r_1^2 h_1 = \pi(10)^2(30) = (100)(30)\pi = 3000\pi$ cm^3.

Since the height of the new cylinder h_2 = 15 cm and the radius r_2 = 20 cm, the volume of the new cylinder is $V_2 = \pi(20)^2(15) = (400)(15)\pi = 6000\pi$ cm^3. Therefore, the volume of the new container is twice the volume of the old container, as shown in general above.

Incorrect Response A: This response might have come from finding the sum of the radius and height of the new container, subtracting it from the sum of the radius and height of the original container, and then cubing the result. (10 cm + 30 cm) – (15 cm + 20 cm) = 5 cm. The cube of this is 125 cm^3.

Incorrect Response B: This response might have come from finding the sum of the radius and height of the new container, and subtracting it from the sum of the radius and height of the original container to give the incorrect response (10 cm + 30 cm) – (15 cm + 20 cm) = 5 cm. This result is then interpreted as 5 cm^3.

Incorrect Response C: This response most likely comes from the idea that doubling the radius and halving the height leaves the volume unchanged. This line of thinking fails to realize that the radius is squared, so doubling the radius causes the square of the radius to increase by a factor of 4.

34. A fuel tank was approximately $\frac{1}{8}$ full. After adding $50 worth of fuel, the tank was approximately $\frac{3}{4}$ full. If the fuel costs *p* dollars per gallon, approximately how many gallons does the tank hold when full?

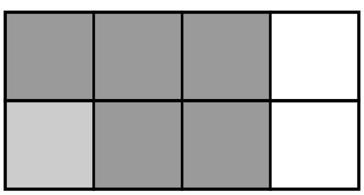
A. $\frac{80}{p}$ gallons

B. $\frac{50}{p}$ gallons

C. 50*p* gallons

D. 80*p* gallons

Correct Response A: One way to solve this problem is to find the cost of gas needed to fill the entire tank and then to determine the number of gallons the tank can hold when full by using the fact that *p* equals dollar price/gallon. To do so, we will model the problem with a diagram.



The diagram represents the gas tank divided into 8 equal sections. Initially, the tank is $\frac{1}{8}$ full, which is represented by the lightly shaded region in the bottom left-hand corner. Since the tank is $\frac{3}{4}$ full after adding the gas, the amount of gas added is given by the five heavily shaded regions. Since this represents $50 worth of gas, each heavily shaded region (one-eighth of a tank) represents $10 worth of gas. It therefore costs $80 to fill the entire tank. The number of gallons needed to fill the tank can then be determined by dividing the price of the gas, $80, by the price of the gas per gallon, or $\frac{80}{p}$. Notice that the units follow the same rules of algebra as do numbers, since $80 \text{ dollars} \div p\,\frac{\text{dollars}}{\text{gallon}} = \frac{80 \text{ dollars}}{1} \bullet \frac{1}{p}\,\frac{\text{gallons}}{\text{dollars}} = \frac{80}{p}$ gallons.

Another more general method of solving this problem is as follows.

If the tank was $\frac{1}{8}$ full before adding the gas and $\frac{3}{4}$ full after adding the gas, then the portion of the tank that was filled by adding gas can be determined as follows:

$$\frac{3}{4} \text{ of a tank} - \frac{1}{8} \text{ of a tank} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8} \text{ of a tank}$$

Let T = cost to fill an entire tank. Then $\frac{5}{8}$ of T equals \$50, or

$$\frac{5}{8}T = 50.$$

Solve for T.

$$\frac{8}{5} \bullet \frac{5}{8}T = \frac{8}{5} \bullet \frac{50}{1}$$

$$T = \$80$$

It costs \$80 to fill the tank. The number of gallons needed to fill the tank can then be determined by dividing the price of the gas, \$80, by the p, price of the gas per gallon, or $\frac{80}{p}$. Again, notice that the units follow the same rules of algebra as do numbers.

Incorrect Response B: This response results from interpreting \$50 as the cost of the gas needed to fill the entire tank. This answer represents the number of gallons of gas needed to fill $\frac{5}{8}$ of the tank, instead of the number of gallons needed to fill the entire tank.

Incorrect Response C: This response results from assuming that the number of gallons of gas can be found by multiplying by p, rather than dividing by p, and from interpreting \$50 as the cost of the gas needed to fill the entire tank.

Incorrect Response D: This response results from correctly finding the cost of the gasoline needed to fill the entire tank, but then finding the number of gallons by multiplying by p, rather than dividing by p.

35. A homeowner is planning to use carpet tiles to cover the floor of a room measuring 9 feet by 10 feet 8 inches. If the carpet tiles are 8 inches wide and 1 foot long and there are no gaps between the tiles as they are placed on the floor, how many carpet tiles will the homeowner need to cover the floor of the room?

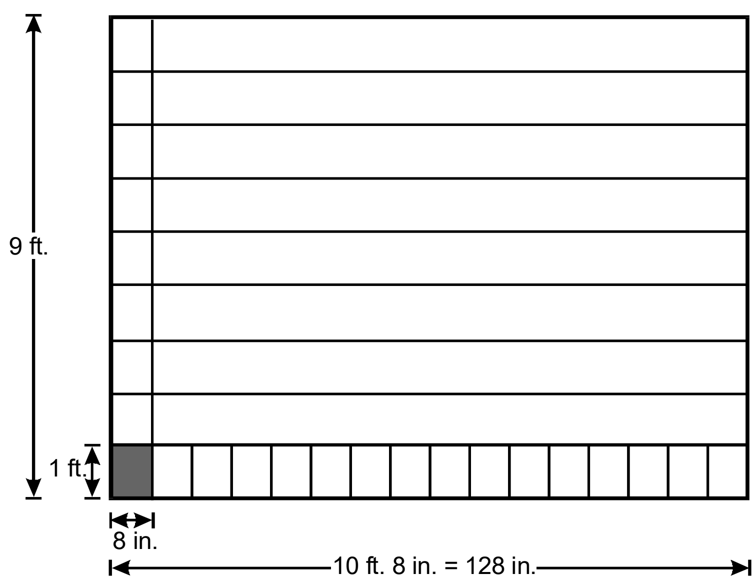
    A. 100

    B. 135

    C. 144

    D. 150

Correct Response C: One way to solve this problem is to sketch a rough diagram of the floor and to examine how the tiles could be placed on the floor. We know that the floor has a width of 9 ft. and a length of 10 ft., 8 in., and each tile is 1 ft. by 8 in.



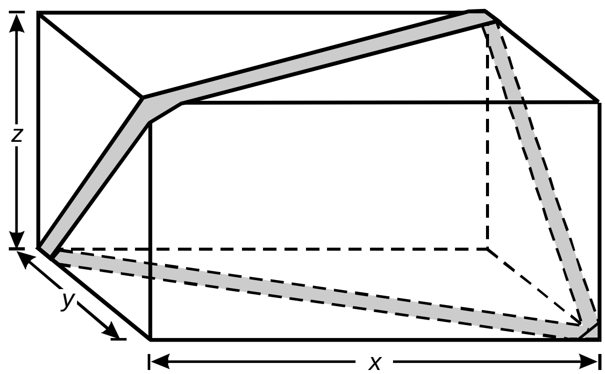
Notice that the only way the tiles could fit on the floor with no gaps is if the tiles are arranged so that the 1 ft. side of a tile corresponds with the 9 ft. width of the floor and the 8 in. side of a tile corresponds to the 10 ft. 8 in. length of the floor. This is because 9 ft. is equal to 108 in. ($9 \text{ ft.} \bullet \frac{12 \text{ in.}}{1 \text{ ft.}} = 108 \text{ in.}$) and 108 is not evenly divisible by 8. On the other hand, 10 ft., 8 in. is equal to 128 inches ($10 \text{ ft} \bullet \frac{12 \text{ in.}}{1 \text{ ft.}} = 120 \text{ in.}$, so 10 ft., 8 in. = 120 in. + 8 in.), which is divisible by 8: $128 \div 8 = 16$. Therefore, 9 tiles will fit along the width of the floor and 16 tiles will fit along the length, so there will be 9 rows of 16 tiles for a total of $9 \bullet 16 = 144$ tiles.

Incorrect Response A: This response could have come from estimating the area of the room in square feet by rounding 10 ft. 8 in. to 11 ft. The area is approximately 99 $ft.^2$, or, rounding up, 100 $ft.^2$ If the estimate of each tile is 1 $ft.^2$, then 100 tiles would be needed, an answer not close to the accurate answer of 144 tiles.

Incorrect Response B: This response could have come from rounding 10 ft. 8 in. down to 10 ft. and estimating the area as 9 ft. • 10 ft., or 90 $ft.^2$. Dividing this value by the area of a tile in square feet gives $90 \text{ ft.}^2 \div \frac{2}{3} \text{ ft.}^2 = 90 \text{ ft.}^2 \cdot \frac{3}{2 \text{ ft.}^2} = 135$. The estimate of 135 tiles needed is too low.

Incorrect Response D: This response could have come from rounding 10 ft. 8 in. to 11 ft., estimating the area of the floor to be approximately 100 $ft.^2$, and then dividing by the area of a tile in square feet to get $100 \text{ ft.}^2 \div \frac{2}{3} \text{ ft.}^2 = 100 \text{ ft.}^2 \cdot \frac{3}{2 \text{ ft.}^2} = 150$. The estimate of 150 tiles needed is too high.

36. **Use the diagram below to answer the question that follows.**



A gift box has dimensions x by y by z. A decorative ribbon is wrapped across the diagonals of the box as shown above. Which of the following expressions represents the approximate total length of the ribbon?

A. $2\left(\sqrt{xy} + \sqrt{yz}\right)$

B. $2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)$

C. $2\left(\sqrt{x^2 + y^2} + \sqrt{y^2 + z^2}\right)$

D. $2\left(\sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2}\right)$

Correct Response C: Finding the length of the ribbon requires determining the length of the diagonals of the faces of the box that are crossed by the ribbon. The length of each diagonal can be determined by recognizing that it is a hypotenuse of a right triangle. According to the Pythagorean theorem, for any right triangle with legs of length a and b, the square of the length of the hypotenuse, c, is equal to the sum of the squares of the lengths of the legs of the triangle, or $c^2 = a^2 + b^2$. Taking the square root of both sides of this equation shows that the length of the hypotenuse is given by $c = \sqrt{a^2 + b^2}$. In this question, the ribbon diagonally crosses four faces of the box.

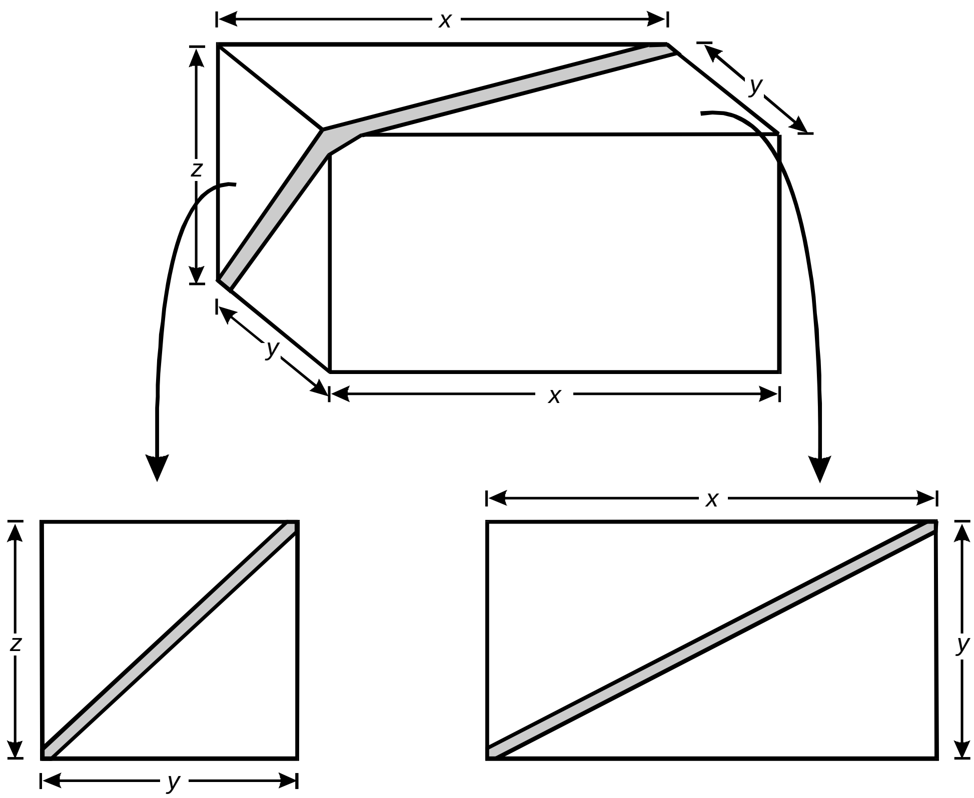


Figure 1 **Figure 2**

Figure 1 shows the left face of the box with a section of the ribbon crossing a diagonal. The right face of the box has the same dimensions and a section of the ribbon crossing a diagonal. Since this rectangular face has edges of length y and length z, the Pythagorean theorem gives the length of the diagonal as

$$\sqrt{y^2 + z^2}.$$

Since the ribbon crosses a diagonal on two congruent faces, the total length of ribbon across the right and left faces is

$$2\sqrt{y^2 + z^2}.$$

Figure 2 shows the top rectangular face with edges of length x and length y. The length of this diagonal is

$$\sqrt{x^2 + y^2}.$$

Since the bottom rectangular face is congruent to the top, the total length of the ribbon across the top and bottom is

$$2\sqrt{x^2 + y^2}.$$

The total length of ribbon is therefore

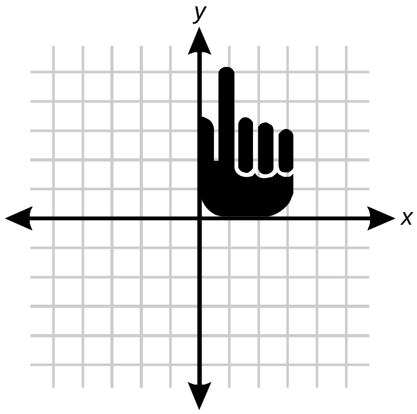
$$2\sqrt{y^2 + z^2} + 2\sqrt{x^2 + y^2} = 2\sqrt{x^2 + y^2} + 2\sqrt{y^2 + z^2} = 2\left(\sqrt{x^2 + y^2} + \sqrt{y^2 + z^2}\right).$$

Incorrect Response A: In this response, the terms *xy* and *yz* represent the area of each face of the box and might have come from mistakenly applying the Pythagorean theorem by finding the product of the edges, rather than the sum of the squares of the edges.

Incorrect Response B: In this response, the terms *xy*, *yz*, and *xz* represent the area of each face of the box and might have come from mistakenly applying the Pythagorean theorem by finding the product of the edges rather than the sum of the squares of the edges. In addition, the expression in B includes an extra term, $\sqrt{zx}$. Note that the front and back faces of the box have edges of length *z* and *x,* but the ribbon does not cross those faces.

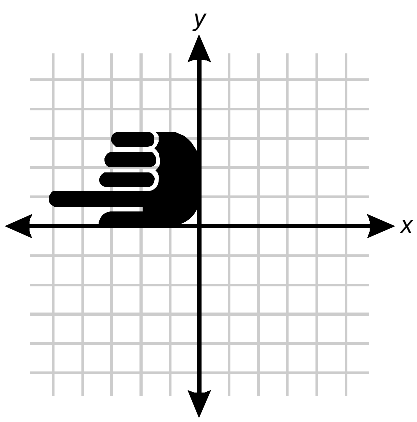
Incorrect Response D: In this response, the Pythagorean theorem is applied correctly, but the expression includes an extra term, $\sqrt{z^2 + x^2}$. This term represents the length of a diagonal of the front or back edge of the box, since those faces have edges of length *z* and *x.* However, the ribbon does not cross the front or back faces of the box.

37. **Use the graph below to answer the question that follows.**

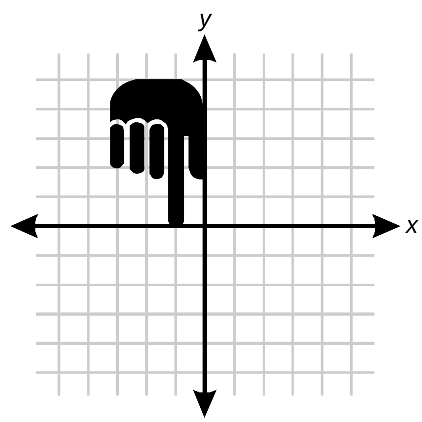


If the image of a pointing hand in the graph above is rotated 180° about the origin and then reflected across the *x*-axis, which of the following graphs will result?

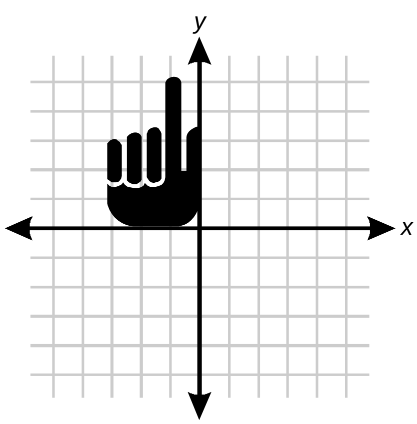
A.



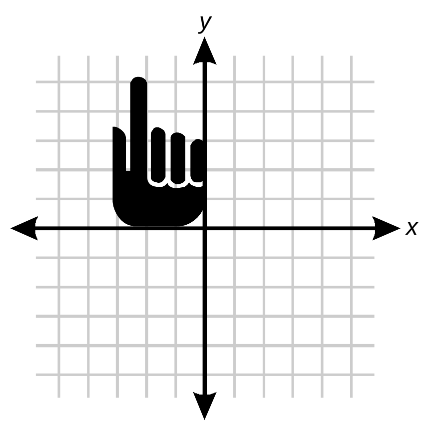
B.



C.



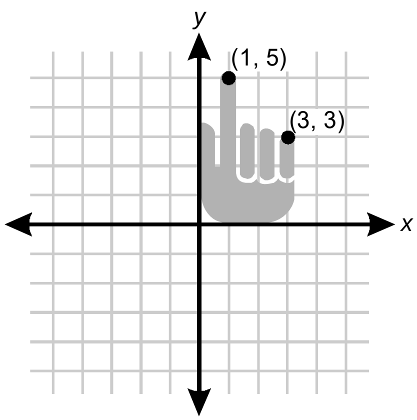
D.



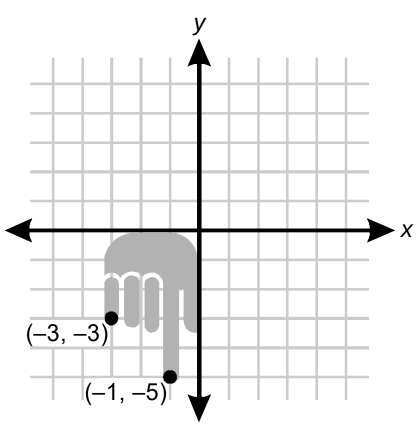
Correct Response C: To answer this question it is necessary to apply two transformations to the figure: first a 180° rotation about the origin and then a reflection across the *x*-axis.

To rotate a figure about a point, in this case the origin, each point on the figure is rotated around the origin. The origin, which is the point with coordinates (0, 0), is the only point that is not rotated. It remains fixed throughout the rotation. A rotation through a positive angle is a rotation in the counterclockwise direction, although in this case it does not matter since the figure is rotated 180°. Rotations are transformations that always result in a figure congruent to the initial figure.

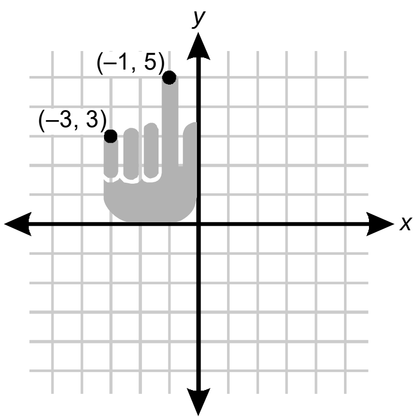
One way to visualize a 180° rotation of this figure is to look at the coordinates of one or more points on the figure. For example, the tip of the longest finger is close to the point (1, 5) and the tip of the shortest finger is close to (3, 3).



You may find it helpful to place your hand in the same orientation and imagine rotating it in the counterclockwise direction by 180° around the origin [i.e., the point (0,0)]. After the hand is rotated through 180°, the tip of the longest finger will have coordinates of (–1, –5) and the tip of the shortest finger will have coordinates (–3, –3). The final rotated figure is shown below.



Next, reflect this rotated image across the *x*-axis. The reflected figure will be a mirror image congruent to the rotated image. The new *y*-coordinate of each point in the reflected figure will be the negative of the original *y*-coordinate (just before reflection) while the *x*-coordinate of each point will remain the same. Since the initial y-coordinates are all negative, the transformed y-coordinates will be positive (the negative of a negative is positive), so the point (–1, –5) is transformed to (–1, 5) and the point (–3, –3) is transformed to (–3, 3). This results in the figure shown below.



~~This results in the figure shown above.~~

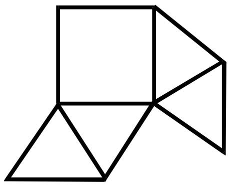
Incorrect Response A: This response could result from rotating the original figure 90° counter-clockwise about the origin. This results in (1, 5) being transformed to (–5, 1) and (3, 3) being transformed to (–3, 3).

Incorrect Response B: This response could result from rotating the original figure 180° about the origin, but instead of being reflected across the x-axis, the figure is translated approximately five units upward to a position above the x-axis. In terms of coordinates, (1, 5) goes to (–1, –5) and then to (–1, 0) while (3, 3) goes to (–3, –3) and then to (–3, 2).

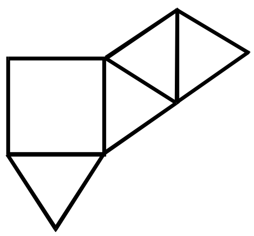
Incorrect Response D: This response results from translating the original figure along the x-axis in the negative x-direction, approximately three units to the left. In terms of coordinates, (1, 5) goes to (–2, 5) and (3, 3) goes to approximately (0, 3).

38. Which of the following nets can be folded to form a square pyramid?

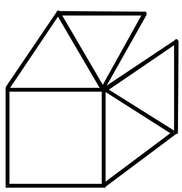
A.



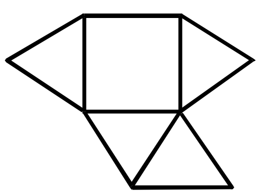
B.



C.

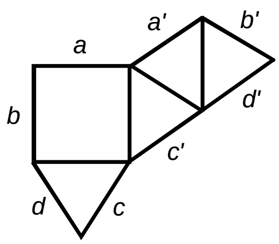


D.

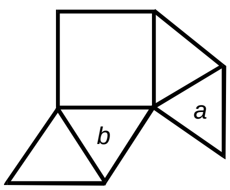


A net is a two-dimensional figure that can be folded into a three-dimensional shape. The question asks which of the nets can be folded into a square pyramid. A square pyramid is a three-dimensional figure that has a square base and four triangular faces. To answer this question, imagine cutting out each plane figure along its perimeter and then folding it along the edges to create a three- dimensional shape. It is often helpful to imagine the square remaining in the plane of the paper and then folding to create a shape either above the plane of the paper or below the plane of the paper. While doing so, note that equivalent 3-D figures result either way.

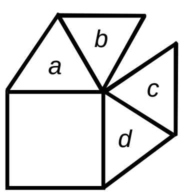
Correct Response B: In the diagram from response B shown below, each edge of the figure is labeled with a letter and its corresponding edge with that same letter with a prime mark. A square pyramid can be formed by folding the net so that corresponding edges are matched. Each of the triangles forms one face of the pyramid and none of the triangles overlap.



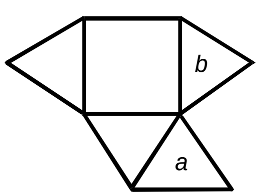
Incorrect Response A: When the net in response A is folded, the triangles labeled *a* and *b* will overlap, resulting in a shape that has a square base but only three faces instead of four. The figure will not be closed. One face of the figure will be missing.



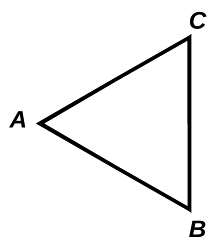
Incorrect Response C: When the net in response C is folded, the triangles labeled *c* and *a* will overlap and triangles *d* and *b* will overlap, resulting in a shape that has a square base and only two faces. The figure will not be closed, since two faces of the figure will be missing.



Incorrect Response D: When the net in response D is folded, the triangles labeled *a* and *b* will overlap, resulting in a shape that has a square base and three faces. This figure will not be closed. One face of the figure will be missing.



39. **Use the figure below to answer the question that follows.**



If equilateral triangle *ABC* above represents one of two congruent halves of a figure that has *AB* as a line of symmetry, then the entire figure is a:

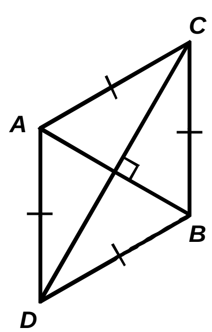
A. triangle.

B. rectangle.

C. prism.

D. rhombus.

Correct Response D: It is given that triangle *ABC* is an equilateral triangle. This means that all of the sides of triangle *ABC* are congruent. It is also given that segment *AB* is a line of symmetry. This means that the other congruent half of the entire figure can be determined by reflecting triangle *ABC* over segment *AB*. To reflect point *C* over a segment *AB*, draw a line from *C* perpendicular to segment *AB*. The reflected point (point *D*) will lie on the perpendicular line such that the distance of point *D* to segment *AB* is equal to the distance of point *C* to segment *AB*. This reflection is shown below. Note that triangle *ABD* is a mirror image of triangle *ABC*.



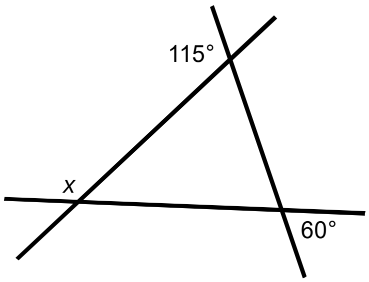
Transformations that are reflections map congruent images to congruent images. This means that triangle *ABD* is also an equilateral triangle and all its sides are congruent. Therefore, *AC* = *CB* = *BD* = *DA* and all sides of quadrilateral *ACBD* are congruent. Since a rhombus is a quadrilateral with all sides congruent, the entire figure *ACBD* is a rhombus. The entire figure cannot be a square since a square has four right angles in addition to four congruent sides.

Incorrect Response A: This response might have come from confusing symmetry with similarity and assuming that triangle *ABC* is similar to the entire figure due to a scaling transformation.

Incorrect Response B: This response might have come from confusing the meaning of a quadrilateral with that of a rectangle. The entire figure cannot be a rectangle since a rectangle has four right angles, each 90°. The original figure was an equilateral triangle with each angle being 60°. Hence, the new figure *ACBD* has 60° angles at *C* and *D*, and 120° angles at *A* and *B*.

Incorrect Response C: This response might have come from thinking that the entire figure could be a three-dimensional shape. However, the entire figure must be a two-dimensional shape since if *ABC* is one half of a figure that has segment *AB* as a line of symmetry, the entire figure must be a two-dimensional figure with both triangles in the same plane.

40. **Use the diagram below to answer the question that follows.**



Three straight lines intersect to form a triangle, as shown above. What is the measure of angle x?

A. 115°

B. 120°

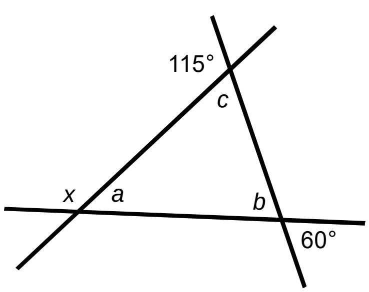
C. 125°

D. 130°

Correct Response C: Label the angles as shown in the triangle below. Denote the measure of an angle (such as $\angle b$) as $m\angle b$. The 60° angle and $\angle b$ are vertical angles. Since vertical angles are congruent, $m\angle b = 60°$.

The 115° angle forms a straight line with $\angle c$ and is therefore supplementary to $\angle c$. Since the sum of the measures of these two angles must equal 180°, $m\angle c + 115° = 180°$, so, $m\angle c = 180° – 115° = 65°$.

The sum of the angles in a triangle equals 180°. Therefore, $m\angle a + m\angle b + m\angle c = 180°$. Using the values for $m\angle b$ and $m\angle c$ found above, it follows that $m\angle a = 180° – (60° + 65°) = 55°$. Finally, the sum of the measure of x and the measure of $\angle a$ equals 180° because they form a straight line. It follows that the measure of x must equal $180° – 55° = 125°$.



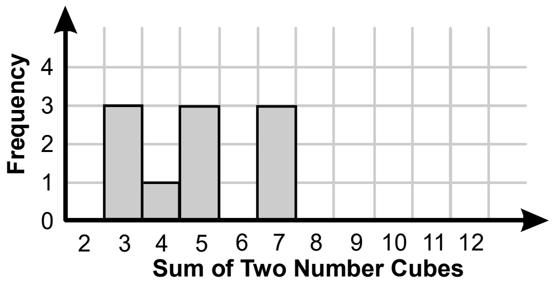
Incorrect Response A: This response may have come from incorrectly assuming that x and the 115° angle have the same measure since they appear to have the same size. It is never a good idea to assume that diagrams are drawn to scale, unless it is specifically stated in the problem. Note that if

$x = 115°$, then $m\angle a = 65°$. Because $m\angle b = 60°$ (vertical angles are congruent), then $m\angle c = 55°$ since the sum of the angles of a triangle equals 180°. This would mean that the given 115° angle would have to be 125°. This contradicts the information given in the problem, so $x \neq 115°$.

Incorrect Response B: This response may have come from incorrectly assuming that the given triangle is equilateral. Note that if $x = 120°$, then the $m\angle c = 60°$ and the given 115° angle would have to be 120°. This contradicts the information given in the problem, so $x \neq 120°$.

Incorrect Response D: This response may have come from assuming that $m\angle a = 50°$, since $\angle a$ looks smaller than $\angle b$ in the diagram. Suppose that $m\angle a = 50°$. Since $m\angle b = 60°$ (vertical angles are congruent), then $m\angle c = 70°$, which would make the given 115° angle have to be 110°. This contradicts the information given in the problem, so $x \neq 130°$.

41. **Use the histogram below to answer the question that follows.**



Two 6-sided number cubes are rolled simultaneously 10 times. The sums are recorded in the histogram shown above. Which of the following statements can be inferred from the histogram?

A. The mean is less than the median by $\frac{1}{10}$.

B. The mean is greater than the median by $\frac{1}{10}$.

C. The mean is less than the median by $\frac{1}{2}$.

D. The mean is greater than the median by $\frac{1}{2}$.

Correct Response A: To answer this question, it is necessary to interpret the data in the graph to find the mean and median of the data since all of the responses compare the mean and the median. The graph is an example of a frequency histogram. Notice that the graph gives the frequency of the sums of the numbers obtained by rolling two number cubes (dice). For example, when the two dice were rolled ten times, the numbers on the dice added to 3 three times, added to 4 only one time, added to 5 three times, added to 7 three times, and added to 2, 6, 8, 9, 10, 11, and 12 zero times. The data set, which is the sum of the numbers obtained on each roll of two dice, is therefore as follows.

3, 3, 3, 4, 5, 5, 5, 7, 7, 7

Notice that there are 10 data points. The mean (arithmetical average) of these data is given by

$$\frac{3+3+3+4+5+5+5+7+7+7}{10} = \frac{49}{10} = 4.9.$$

This could also be found as follows.

$$\frac{[3(3)+1(4)+3(5)+3(7)]}{10} = 4.9$$

The median is found by ordering all of the data points from least to greatest and then determining the middle value in the list. The ordered list of data points is

3, 3, 3, 4, , , 5, 7, 7, 7

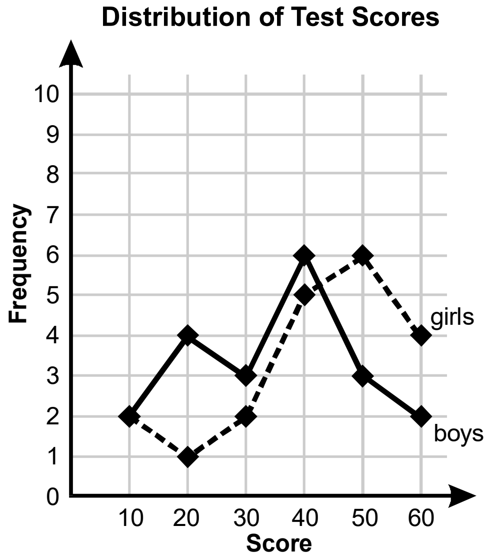
When the number of items in the list is even, in this case 10, the median value is the mean (arithmetical average) of the middle two values. For the data in this problem, the two middle values are the fifth and sixth values (shaded), each of which is 5. The median value is $(5 + 5) \div 2$, or 5. The difference between the median and the mean is $5 - 4.9 = 0.1$, or $\frac{1}{10}$. The mean is less than the median by $\frac{1}{10}$.

Incorrect Response B: This response may have come from confusing the median and the mean and using the value of 5 for the mean and the value of 4.9 for the median.

Incorrect Response C: This response might have come from incorrectly interpreting the bars on the graph in such a way that the median is the value halfway between 3 and 7 or $(3 + 7) \div 2 = 5$, and the mean is the average of 4 and 5 or 4.5. Then the mean would be less than the median by 0.5.

Incorrect Response D: This response may have come from incorrectly reading the graph and interpreting the mean as 5, since it is the "middle bar," and then interpreting the median as 4.5, the average of 4 and 5. Then the mean would be greater than the median by 0.5.

42. **Use the graph below to answer the question that follows**.



The graph above shows the distribution of scores on a test with possible scores of 10, 20, 30, 40, 50, and 60. The minimum passing score was 40. 20 girls and 20 boys took the test. The percentage of girls passing the test was how much greater than the percentage of boys passing the test?

A. 25%

B. 20%

C. 15%

D. 10%

Correct Response B: Answering this question requires comparing the percentage of girls who passed the test with the percentage of boys who passed the test. To determine the percentage of girls who passed the test it is necessary to know how many girls took the test and how many obtained a score of 40 or above, since the passing score is 40. We are told that 20 girls took the test, so we need to use the graph to find the number of girls who scored 40 or above. Note that the graph gives the frequency, or the number of girls or boys who obtained a given score. The girls' scores are indicated by the points on the dotted line. From the graph, 5 girls scored 40, 6 girls scored 50, and 4 girls scored 60. The total number of girls who passed is therefore 5 + 6 + 4 = 15 girls. Since 20 girls took the test, the percentage of girls passing is $\frac{15}{20}$ = 75%.

A similar procedure can be applied to find the number of boys who passed the test, and therefore the percentage of boys who passed. The boys' scores are represented by the points on the solid line. Note that 6 boys obtained a 40, 3 boys obtained a 50, and 2 boys obtained a 60, so 6 + 3 + 2 = 11 boys out of the 20 passed the test. The percentage passing is $\frac{11}{20}$ = 55%. The difference in the percentage of girls passing the test and the percentage of boys passing the test is 75% – 55% or 20%.

Incorrect Response A: This response might have come from incorrectly interpreting the passing score as a score greater than 40. In this case, 10 girls obtained a score of 50 or 60 and $\frac{10}{20}$ or 50% of the girls passed. Five boys obtained a score of 50 or 60 and $\frac{5}{20}$, or 25% of the boys passed. This leads to a difference of 25%.

Incorrect Response C: This response might have come from calculating 15 as the number of girls who obtained a score of 40 or higher, and then interpreting this number as a percentage, without attempting to calculate the difference in percentage.

Incorrect Response D: This response might have come from using only the last data points, which are the number of girls and boys who obtained a score of 60 on the test. This incorrectly leads to $\frac{4}{20}$, or 20% of the girls passing and $\frac{2}{20}$ or 10% of the boys passing for a difference of 10%.

43. **Use the table below to answer the question that follows.**

	Milk Shake Flavor Preference		
Age Range	**Vanilla**	**Chocolate**	**Strawberry**
10–19	7	30	15
20–29	10	16	14
30–39	8	14	12
40–49	9	17	9

A marketing company conducted a survey to determine milk shake flavor preferences among five different age groups. Based on the data in the table, what is the probability that a randomly chosen 35-year-old customer will prefer a strawberry-flavored milk shake?

A. $\frac{1}{7}$

B. $\frac{1}{6}$

C. $\frac{6}{25}$

D. $\frac{6}{17}$

Correct Response D: The question asks for the probability that a randomly chosen 35-year-old customer will prefer a strawberry-flavored milk shake. To determine this probability, it is necessary to first interpret the table. The table gives the number of people in a given age range who prefer each flavor of milk shake. For example, in the age range 30–39, 8 people preferred vanilla, 14 people preferred chocolate, and 12 people preferred strawberry. The total number of 30–39 year olds surveyed is 8 + 14 + 12 = 34.

The probability of an event is the number of favorable outcomes divided by the total number of possible outcomes. In this case, the number of favorable outcomes is the number of 30–39 year olds who prefer strawberry milk shakes and the total number of possible outcomes is the total number of 30–39 year olds surveyed. The probability that a randomly selected 35-year-old will prefer a strawberry-flavored milk shake is therefore given by the following fraction.

$$\frac{\text{number of 30–39 years old who prefer strawberry}}{\text{total number of 30–39 year olds}} = \frac{12}{34} = \frac{6}{17}$$

Incorrect Response A: This response may come from a misunderstanding of the favorable outcomes and the total number of outcomes. The table has 4 age ranges and 3 flavor choices. Interpreting the total number of outcomes as the total number of categories or 3 + 4 = 7, and the number of favorable outcomes as 1 category representing strawberry, gives $\frac{1}{7}$.

Incorrect Response B: This response correctly uses 12 for the number of favorable outcomes. The total number of outcomes is incorrectly computed as the union of the number of 30–39 year olds surveyed (8 + 14 + 12) and the additional number of people surveyed who prefer strawberry (15 + 14 + 9). Note that the 12 is not repeated. This would give $\frac{12}{8 + 14 + 12 + 15 + 14 + 9} = \frac{12}{72} = \frac{1}{6}$. This response answers the question as to what is the probability that a randomly chosen person from either the 30–39 age group or the group of people who prefer strawberry milk shakes is in both the 30–39 age group and the group of people who prefer strawberry milk shakes.

Incorrect Response C: This response may come from incorrectly thinking that you are asked for the probability that a randomly chosen person who prefers strawberry milk shakes is in the 30–39 age group. The number of favorable outcomes is the number of 30–39 year olds who prefer strawberry milk shakes or 12, as found above in the correct response. The total number of outcomes is equal to the total number of people who prefer strawberry or 15 + 14 + 12 + 9 = 50. This results in a probability of $\frac{12}{50}$, or $\frac{6}{25}$.

44. A child has a set of blocks, of which 4 are square, 5 are round, and 6 are triangular. The child randomly picks a block from the set and gives it to his sister. The child then randomly picks one more block. What is the probability that the first block was round and the second block was triangular?

    A. $\frac{1}{9}$

    B. $\frac{2}{15}$

    C. $\frac{1}{7}$

    D. $\frac{11}{15}$

Correct Response C: The problem asks to determine the probability that the first block was round and the second block was triangular. Notice that these two events are dependent; that is, the probability of the second event depends on the first event.

The total number of blocks in the original set is 4 + 5 + 6 = 15. Of those, 5 are round. Therefore,

$$\text{probability of round block} = \frac{\text{number of round blocks}}{\text{total number of blocks}} = \frac{5}{15}.$$

One round block was chosen first, without replacement. We know that the round block wasn't returned to the set because the question states, "The child randomly picks a block from the set *and gives it to his sister.*" Since one round block was chosen and not returned to the set, there are now only 14 blocks remaining in the set. Since 6 of them are triangular, the probability of randomly selecting a triangular block is

$$\text{probability of triangular block} = \frac{\text{number of triangular blocks}}{\text{total number of blocks remaining}} = \frac{6}{14}.$$

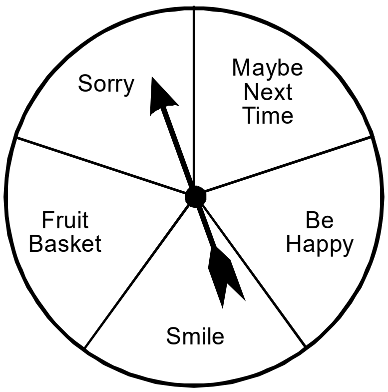
The probability of first selecting a round block and then selecting a triangular block is given by the product of these two probabilities, which is $\frac{5}{15} \bullet \frac{6}{14} = \frac{30}{210} = \frac{1}{7}$.

Incorrect Response A: This response might have come from noting that there are 3 different types of blocks in the set and incorrectly concluding that the probability of picking a round block is $\frac{1}{3}$, and the probability of picking a triangular block is also $\frac{1}{3}$. The probability of first picking a round block and then picking a triangular block then becomes the product of the two probabilities, or $\frac{1}{3} \bullet \frac{1}{3} = \frac{1}{9}$.

Incorrect Response B: This response might have come from assuming that since there are a total of 15 blocks, the probability of picking the first block is equal to $\frac{1}{15}$ and the probability of picking the second block is also $\frac{1}{15}$. The probability of first picking a round block and then picking a triangular block is then assumed to be found by adding the probabilities instead of multiplying: $\frac{1}{15} + \frac{1}{15} = \frac{2}{15}$.

Incorrect Response D: This response might have come from correctly determining the probability of picking a round block to be $\frac{5}{15}$, but determining the probability of picking a triangular block as $\frac{6}{15}$ and then finding the sum of the two probabilities to get $\frac{5}{15} + \frac{6}{15} = \frac{11}{15}$. This value is actually equal to the probability of randomly picking either a round block or a triangular block on the first selection from the set of blocks.

45. **Use the spinner below to answer the question that follows.**



The host of a party tells her guests that every time the spinner above lands on the section labeled "Fruit Basket," a guest will win a large basket of fruit. If the 180 guests at the party each spin the spinner once, what is the best estimate of the number of fruit baskets that the host will be giving away?

A. 7

B. 14

C. 36

D. 72

Correct Response C: Each time the spinner is spun, it has an equal chance of landing in any of the 5 sections. Since only 1 section is labeled "Fruit Basket," every time 5 guests spin the spinner, on average only 1 guest will win a fruit basket. Since there are 180 guests, on average it can be expected that $180 \div 5 = 36$ guests will win fruit baskets. Notice that this value is also equal to the product of the probability of obtaining a "Fruit Basket" on one spin and the total number of spins or $\frac{1}{5} \cdot 180 = 36$.

Incorrect Response A: This response might have come from confusing the idea of the permutation of 5 objects, which would equal $5 \times 4 \times 3 \times 2 \times 1 = 5 \times 4 \times 3 \times 2$, adding instead of multiplying, and then assuming that the total number of winners is given by one-half of $(5 + 4 + 3 + 2)$ or $\frac{1}{2} \cdot 14 = 7$. The one-half might have come from thinking about 180 as being half of 360, since there are 360° in a circle.

Incorrect Response B: This response might have come from confusing the idea of the permutation of 5 objects, which would equal $5 \times 4 \times 3 \times 2 \times 1 = 5 \times 4 \times 3 \times 2$, adding instead of multiplying, and then assuming that the total number of winners is given by the sum of $5 + 4 + 3 + 2 = 14$.

Incorrect Response D: This response might have come from assuming that the average number of guests who win is equal to the measure of the central angle of one of the sections of the circle. Since there are 360° in a circle, the measure of one central angle is $360° \div 5 = 72°$. The degree measure was then ignored.